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Transient Analysis of Single Server Queueing system with Loss and Feedback

S. Shanthi Sivanandam[1*], A. Muthu Ganapathi Subramanian[2], Gopal Sekar[3]

[1,2,3]Department of Mathematics, Kanchi Mamunivar Centre for Post Graduate Studies, Pondicherry, India

**Corresponding Author: sivashanthi00@gmail.com, Tel.: +91-9751600309*

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Abstract — Consider a single server queueing system with Loss and Feedback in which customers arrive in a Poisson process with arrival rate λ and service time follows an exponential distribution with parameter μ. If the server is free at the time of an arrival of a customer, the arriving customer begins to be served immediately by the server and satisfied customer leaves the system with probability (1-q) after the service completion and dissatisfied customers will join the queue with probability q to get service once again. This is called Feedback in queueing terminology. If the server is busy, then the arriving customer will join the queue with probability p in front of service station. This is called Loss in queueing terminology. In this paper, we have derived the closed form solutions of time dependent probabilities of the single server queueing systems with Loss and Feedback. The corresponding Transient distributions have been obtained. We also obtain the time dependent performance measures of the systems.

Keywords — Loss and Feedback - Single Server - Steady State Probabilities –System performance measures- Transient Probability Distributions.

I. INTRODUCTION

During the past few years a number of interesting and innovative research papers have appeared in the literature that discuss the Transient behaviour for the queue length of the single server queueing system at time t. This queue has attracted the attention of many researchers who have proposed its solution by a variety of techniques and have obtained different types of solutions. The main objective of this paper is to analysis the Transient behaviour of Single server queueing system with Loss and Feedback. Parthasarathy have studied a simple approach of a Transient solution to an M/M/1Queue [12]. The Transient behaviour of the M/M/1/N queue for a general N has been discussed by Takacs [18] and Morse [11]. Bailey [7] used Generating functions method to analyse the Transient behaviour of a simple queue. Takacs [19] introduced the concept of Feedback queues. Disney R.L, Gilles R and D'Avignon [8, 9] have studied Queues with State Dependent Feedback. Abate et al. [1, 2] have studied the Transient behaviour of M/M/1 queue using Laplace transforms. Parthasarathy and Lenin [13] used Continued fractions to analyses the Transient behaviour of birth death processes. Sharma and Bunday [15] have investigated the Transient behaviour of M/M/1 queue and have obtained the state probabilities in closed form.
Parthasarathy and Selvaraju [14] have analyzed the Transient behaviour of an M/M/1 queue with loss. Tarabia and et al. [20] have studied the exact Transient solutions to non-empty Markovian queues by using the Power series technique. Thangaraj and Vanitha [21] have considered the Transient analysis of M/M/1 queue with Bernoulli Feedback. Ayyapan, G., Muthu Ganapathi Subramanian, A. and Gopal Sekar [5] have discussed M/M/1 Retrial Queueing System with Loss and Feedback. Sharma S.K. and Kumar R. [16] have discussed Markovian Feedback Queues. Kaczynski WH, Leemis LM and Drew JH [10] have analyzed the Transient behaviour. Ayyappan G and Shyamala S [6] have analyzed the Time Dependent solution of $M^{[X]}/G/1$ Queueing model with Bernoulli vacation and Balking. Singla.N and Garg PC [17] have studied the Transient and Numerical solutions of Feedback. Ammar SI [3] has examined the Transient behaviour of a Two Heterogeneous servers queue with impatient behaviour. Recently, Ammar[4] has examined the Transient solution of an M/M/1 Vacation queue with a Waiting server and impatient customers.

II. MODEL DESCRIPTION

Consider a single server queueing system with Loss and Feedback in which customers arrive in a Poisson process with arrival rate λ and service time follows an exponential distribution with parameter μ. If the server is free at the time of an arrival of a customer, the arriving customer begins to be served immediately by the server and satisfied customer

leaves the system with probability (1-q) after the service completion and dissatisfied customers will join the queue with probability q to get service once again. This is called Feedback in queueing terminology. If the server is busy, then the arriving customer will join the queue with probability p in front of service station. This is called Loss in queueing terminology.
Let N (t) is random variable which represents the number of customers in the system at time t.
The random process is described as

{N (t) / N (t) = 0, 1, 2, 3,}

$p_n(t)$ - Represents the time dependent probability that there are n customers in the system at time t.

$p_0(t)$ - Represents the time dependent probability that there are no customers in the system at time t.

III. ANALYSIS OF TRANSIENT PROBABILITY

The governing differential – difference equations of the single server queueing system are given by means of the chapman – Kolmogorov equations

$$p_0'(t) = -(\lambda p) p_0(t) + \mu(1-q) p_1(t) \quad (1)$$

$$p_n'(t) = -(\lambda p + \mu(1-q)) p_n(t) + (\lambda p) p_{n-1}(t) + \mu(1-q) p_{n+1}(t)$$

for n = 1, 2, 3,…… (2)

In this section, the transient system size probabilities are obtained by using the Laplace transform and Generating functions technique.

Define $p_n^* = \int_0^t p_n(t) e^{-st} dt$ for n = 0, 1, 2, …

Apply the Laplace transform to the system of equations (1) and (2), we get

$$(s + \lambda p) p_0^* = 1 + (1-q)\mu p_1^* \quad (3)$$

$$(s + \lambda p + (1-q)\mu) p_n^* = \lambda p \, p_{n-1}^* + (1-q)\mu p_{n+1}^* \quad (4)$$

for n = 1,2,3,…

Theorem 1:
The transient probabilities that the number of customers in the system at time t are given by

$$p_0(t) = e^{-(\lambda p + (1-q)\mu)t} \sum_{k=0}^{\infty} \beta^{-k} [I_k(\alpha t) - I_{k+2}(\alpha t)]$$

$$p_n(t) = e^{-(\lambda p + (1-q)\mu)t} \sum_{k=0}^{\infty} \beta^{n-k} [I_{n+k}(\alpha t) - I_{n+k+2}(\alpha t)]$$

for n = 1,2,3,…

Proof:

Taking summation from n = 1 to ∞ for equation (4) and using the generating function

$p^*(z,s) = \sum_{n=0}^{\infty} p_n^* z^n$, we get

$$p^*(z,s) = \frac{(1-q)\mu p_0^*(1-z) - z}{\lambda z^2 - z(s + \lambda p + (1-q)\mu) + (1-q)\mu} \quad (5)$$

$$\lambda z^2 - z(s + \lambda p + (1-q)\mu) + (1-q)\mu = 0 \quad (6)$$

Let w_0^* and w_1^* be the roots of (6)

$$w_0^* = \frac{w - \sqrt{w^2 - \alpha^2}}{2\lambda p}, \quad w_1^* = \frac{w + \sqrt{w^2 - \alpha^2}}{2\lambda p}$$

where

$w = s + \lambda p + \mu(1-q)$, $\alpha = 2\sqrt{\lambda p(1-q)\mu}$, $\beta = \sqrt{\frac{\lambda p}{(1-q)\mu}}$ by Rouche's theorem

$|w_0^*| < 1$ and $|w_1^*| > 1$

$\because$ w_0^* lies inside the disc and LHS of (5) converges,

Cancellation of $(z - w_0^*)$ with numerator of (5). We get

$$(1-q)\mu p_0^*(1 - w_0^*) - w_0^* = 0$$

$$p_0^* = \frac{1}{(1-q)\mu} \sum_{k=0}^{\infty} w_0^{*\,k+1} \quad (7)$$

We know that

$$L^{-1}\left(\frac{\alpha^n}{\left(s + \sqrt{s^2 - \alpha^2}\right)^n} \right) = \frac{n}{t} I_n(\alpha t) \quad (8)$$

$$\frac{n}{t} I_n(\alpha t) = \frac{\alpha}{2} [I_{n-1}(\alpha t) - I_{n+1}(\alpha t)]$$

(9)

Where $I_n(.)$ is the Modified Bessel function of order n.

Apply inverse Laplace transform (7), we get

$$p_0(t) = e^{-(\lambda p + (1-q)\mu)t} \sum_{k=0}^{\infty} \beta^{-k} [I_k(\alpha t) - I_{k+2}(\alpha t)] \quad (10)$$

Equation (10) represents the transient probability that there are no customers in the system at time t.
Equation (5) can be written as

$$p^*(z,s) = \frac{w_1^* p_0^*}{(w_1^* - z)} \quad (11)$$

From equation (11),

$$p_n^* = p_0^* \left(\frac{1}{w_1^*} \right)^n \quad (12)$$

$$p_n^* = \left(\frac{\beta^{2n}}{(1-q)\mu}\right)\sum_{k=0}^{\infty}\left(w_0^*\right)^{n+k+1} \qquad (13)$$

Apply inverse Laplace transform to the equation (13), we get

$$p_n(t) = e^{-(\lambda p+(1-q)\mu)t}\sum_{k=0}^{\infty}\beta^{n-k}[I_{n+k}(\alpha t) - I_{n+k+2}(\alpha t)] \qquad (14)$$

for n = 1,2,3,…

Equation (14) represents the transition probability that there are n customers in the system at time t.

The time dependent solutions for the number of customers in the system at time t are given by

$$p_0(t) = e^{-(\lambda p+(1-q)\mu)t}\sum_{k=0}^{\infty}\beta^{-k}[I_k(\alpha t) - I_{k+2}(\alpha t)]$$

$$p_n(t) = e^{-(\lambda p+(1-q)\mu)t}\sum_{k=0}^{\infty}\beta^{n-k}[I_{n+k}(\alpha t) - I_{n+k+2}(\alpha t)]$$

for n = 1,2,3,…

REMARK

1. As $p \to 1$ and $q \to 0$, the equations (10) and (14) coincides with Parthasarathy's [12] new approach for transient behavior of M/M/1 model.
2. As $q \to 0$, the equations (10) and (14) coincides with P.R. Parthasarathy and N. Selvaraju [14].

IV. ANALYSIS OF STEADY STATE PROBABILITY

Theorem 2:

The steady state probabilities that the number of customers in the system are given by

$$p_0 = 1-\rho$$

$$p_n = p_0\rho^n \qquad \text{for } n = 1,2,3....$$

Proof:

The steady state probabilities can be obtained by using the Final value theorem on Laplace transform.

$$p_0^* = \frac{1}{(1-q)\mu}\left(\frac{w_0^*}{1-w_0^*}\right)$$

$$p_0 = \lim_{s\to 0} sp_0^* = \lim_{s\to 0} s\frac{1}{(1-q)\mu}\left(\frac{w_0^*}{1-w_0^*}\right)$$

As $s \to 0$, $w_0^* \to 1$, $w_0^{*'} \to \frac{-1}{(1-q)\mu - \lambda p}$

$$= \lim_{s\to 0} s\frac{1}{(1-q)\mu}\left(\frac{w_0^*}{1-w_0^*}\right) = \lim_{s\to 0}\frac{1}{(1-q)\mu}\left(\frac{1}{-w_0^{*'}}\right)$$

$$= 1 - \frac{\lambda p}{(1-q)\mu} = 1-\rho$$

$$p_0 = 1-\rho \quad \text{where } \rho = \frac{\lambda p}{(1-q)\mu} \qquad (15)$$

Equation (15) represents the steady state probability that there is no customer in the system.

From equation (12), we get

$$p_n^* = p_0^*\left(\frac{\lambda p w_0^*}{(1-q)\mu}\right)^n$$

$$p_n = \lim_{s\to 0} sp_n^* = \lim_{s\to 0} sp_0^*\left(\frac{\lambda p w_0^*}{(1-q)\mu}\right)^n = p_0\rho^n$$

$$p_n = p_0\rho^n \text{, where } \rho = \frac{\lambda p}{(1-q)\mu} \qquad (16)$$

The equation (16) represents the steady state probabilities of n customers in the system.

V. SYSTEM PERFORMANCE MEASURES

In this section, we will list some important performance measures along with their formulas. These measures are used to bring out the qualitative Transient behaviour of the queueing model under study. Numerical study has been dealt in very large scale to study the following measures.

1. Probability that the server is idle at time t = $P_0(t)$
2. Probability that the server is busy at time t = $\sum_{n=1}^{\infty} P_n(t)$
3. M(t) = Average number of customers in the system at time t = $\sum_{n=1}^{\infty} nP_n(t)$
4. m(t) = Average number of customers in the queue at time t = $\sum_{n=2}^{\infty}(n-1)P_n(t)$
5. V(t) = Variance of the number of customers in the system at time t
   $$= \sum_{n=1}^{\infty} n^2P_n(t) - \left(\sum_{n=1}^{\infty} nP_n(t)\right)^2$$
6. W(t) = Average waiting of a customer in the system at time t = M(t)/λ
7. w(t) = Average waiting of a customer in the queue at time t = m(t)/λ

VI. NUMERICAL COMPUTATIONS

The system performance measures and Transient probabilities of this model have been done and expressed in the form of tables, which are shown below for various values λ, μ, p, q and t.

Table 1,Table 2 and Table 3shows Transient probabilities of number of customers in the system for various values of λ, μ and t.We infer the following

- As the value of t increases the Transient probabilities $p_n(t) \to p_n$ where p_n is the steady state probability that there are *n* customers in the system.
- The sequence $\{p_n(t)\} \to 0$ as $n \to \infty$ for all values of t

Table 1: Transient probability distribution of number of customers in the system for various values of t, when λ =1, μ=10, p=0.3 and q=0.4

t	P_0 (t)	P_1 (t)	p_2 (t)	p_3 (t)	p_4 (t)	p_5 (t)
1	0.9489	0.0487	0.0024	0.0000	0.0000	0.0000
2	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
3	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
4	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
5	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
6	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
7	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
8	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
9	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000
10	0.9500	0.0475	0.0024	0.0001	0.0000	0.0000

Table 2: Transient probability distribution of number of customers in the system for various values of t, when λ =5, μ=10, p=0.3 and q=0.4

t	P_0 (t)	P_1 (t)	p_2 (t)	p_3 (t)	p_4 (t)	p_5 (t)
1	0.7566	0.1863	0.0444	0.0101	0.0021	0.0004
2	0.7507	0.1874	0.0467	0.0115	0.0028	0.0007
3	0.7501	0.1875	0.0468	0.0117	0.0029	0.0007
4	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
5	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
6	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
7	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
8	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
9	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007
10	0.7500	0.1875	0.0469	0.0117	0.0029	0.0007

Table 3: Transient probability distribution of number of customers in the system for various values of t, when λ =9, μ=10, p=0.3 and q = 0.4

t	P_0 (t)	P_1 (t)	p_2 (t)	p_3 (t)	p_4 (t)	p_5 (t)
1	0.5790	0.2527	0.1051	0.0411	0.0149	0.0050
2	0.5574	0.2494	0.1104	0.0481	0.0206	0.0086
3	0.5524	0.2482	0.1111	0.0495	0.0219	0.0096
4	0.5509	0.2478	0.1113	0.0499	0.0223	0.0099
5	0.5504	0.2476	0.1114	0.0500	0.0225	0.0101
6	0.5501	0.2475	0.1114	0.0501	0.0225	0.0101
7	0.5501	0.2475	0.1114	0.0501	0.0225	0.0101
8	0.5500	0.2475	0.1114	0.0501	0.0225	0.0101
9	0.5500	0.2475	0.1114	0.0501	0.0226	0.0101
10	0.5500	0.2475	0.1114	0.0501	0.0226	0.0101

Table 4, Table 5 and Table 6 shows Transient System performance measures for various values of λ, μ and t.We infer the following

- $p_{idle}(t)$ decreases as arrival rate λ increases for all values of t
- $p_{busy}(t)$ increases as arrival rate λ increases for all values of t
- $W(t)$ and $w(t)$ increases as arrival rate λ increases for all values of t
- As t increases, $p_{idle}(t) \to p_{idle}$, $p_{busy}(t) \to p_{busy}$, $M(t) \to L_s, m(t) \to L_q$, $W(t) \to W_s, w(t) \to W_q$

Table 4: System performance measures for various values of t, when λ = 1, μ = 10, p = 0.3 and q = 0.4

t	P_{idle}(t)	P_{busy}(t)	M (t)	m (t)	W(t)	w (t)
1	0.9489	0.0510	0.0536	0.0025	0.0536	0.0025
2	0.9500	0.0499	0.0526	0.0026	0.0526	0.0026
3	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
4	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
5	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
6	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
7	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
8	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
9	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026
10	0.9500	0.0500	0.0526	0.0026	0.0526	0.0026

Table 5: System performance measures for various values of t, when λ = 5, μ = 10, p = 0.3 and q = 0.4

t	P_{idle}(t)	P_{busy}(t)	M (t)	m (t)	W(t)	w (t)
1	0.7566	0.2434	0.3165	0.0731	0.0633	0.0146
2	0.7507	0.2493	0.3314	0.0820	0.0663	0.0164
3	0.7501	0.2499	0.3331	0.0831	0.0666	0.0166
4	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167
5	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167
6	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167
7	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167

8	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167
9	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167
10	0.7500	0.2500	0.3333	0.0833	0.0667	0.0167

Table 6: System performance measures for various values of t, when λ = 9, μ = 10, p = 0.3 and q = 0.4

t	$P_{idle}(t)$	$P_{busy}(t)$	M (t)	m (t)	W(t)	w (t)
1	0.5790	0.4210	0.6845	0.2635	0.0761	0.0293
2	0.5574	0.4426	0.7764	0.3338	0.0863	0.0371
3	0.5524	0.4476	0.8029	0.3553	0.0892	0.0395
4	0.5509	0.4491	0.8121	0.3630	0.0902	0.0403
5	0.5504	0.4496	0.8157	0.3660	0.0906	0.0407
6	0.5501	0.4499	0.8171	0.3673	0.0908	0.0408
7	0.5501	0.4499	0.8177	0.3678	0.0909	0.0409
8	0.5500	0.4500	0.8180	0.3680	0.0909	0.0409
9	0.5500	0.4500	0.8181	0.3681	0.0909	0.0409
10	0.5500	0.4500	0.8181	0.3681	0.0909	0.0409

VII. CONCLUSION

In this paper, Loss and Feedback queueing model is considered in which customers, whose arrival times are governed by a Markovian arrival process and exponential service times. The closed form solutions of Transient probability distribution and system performance measures are determined analytically.

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Authors Profile

Dr. A.Muthu Ganapathi Subramanian is an Associate Professor of Mathematics, KMCPGS,Pondicherry and specilised in Rerial queueing models. He has published more than 50 papers in various peer Reviewed journals and UGC approved journals and also published two books in Retrial queueing theory.He is guiding four research scholasr towards Ph.d program and also produced many M.phil research scholars.

Dr.Gopal sekar is an Associate Professor of Mathematics, KMCPGS,Pondicherry and specilised in Rerial queueing models. He has published more than 40 papers in various peer Reviewed journals and UGC approved journals. He has produced many M.phil research scholars.

Mrs.S. Shanthi is a full time Research scholar and doing Ph.D program under the guidance of Dr. Muthu Ganapathi Subramanian.