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On The Strong Edge Monophonic Number of Graphs

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***Abstract*—** Abstract For a connected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ of order at least two, a set $\mathbf{S}$ of vertices of $\mathbf{G}$ is a Strong edge Monophonic set if every edge of $\mathbf{G}$ is contained in a fixed monophonic path between any pair of vertices of $\mathbf{S}$. The minimum cardinality of the strong edge monophonic set is the strong edge monophonic number of $\mathbf{G}$ denoted my $\mathbf{Sm_1}(\mathbf{G})$. In this paper, certain general properties of the strong edge monophonic sets are studied. Also the strong monophonic number of some families of graph are determined.

***Keywords*—** Monophonic set, Strong monophonic set, Edge monophonic set, Monophonic distance.

I. INTRODUCTION

By a Graph $G = (V(G), E(G))$ we mean a finite undirected connected graph without loops or multiple edges. For vertices u and v in a connected graph G , the distance $d(u, v)$ is the length of the shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called $u - v$ geodesic. A set $S \subseteq V(G)$ is called a geodetic set if all the vertices of G should lie in the $x - y$ geodesic of the vertices of S . The minimum cardinality of the geodetic set is the geodetic number of G.

The Strong geodetic problem is a variation of the geodetic problem. It is defined as follows.[9] Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V(G)$ is a strong geodetic set if for each pair of vertices $x, y \subseteq S$ $x \neq y$, let $\tilde{g}(x, y)$ be a selected fixed shortest path between x and y . Then

$$\tilde{I}(s) = \{\tilde{g}(x, y): x, y \in S\} \qquad (1)$$

and $V(\tilde{I}(S)) = \cup_{\tilde{p}\in\tilde{I}}(S)\ V(\tilde{P})$. If $V(\tilde{P}) = V$ for some $\tilde{I}(S)$, then the set S is called a strong geodetic set. The minimum cardinality of a strong geodetic set is the strong geodetic number. The Strong geodetic problem is NP-Complete for general graphs.

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chord less path. A set $S \subseteq (V(G))$ of a graph G is a monophonic set of G if each vertex of G lies on a $u - v$ monophonic path in G for some $u, v \in S$. A set $S \subseteq V(G)$ is a edge monophonic set of G if each edge of G lies on a $u - v$ monophonic path of G for $u, v \in E(G)$. The minimum cardinality of the edge monophonic set is the edge monophonic number. Let $G = (V(G), E(G))$ be a non-trivial connected graph. Let $S \subseteq V(G)$, then for each pair of vertices $u, v \in S$, $u \neq v$, let $\tilde{P}(u, v)$ be the selected fixed monophonic path between u and v . Then we set

$$\tilde{J}(S) = \{\tilde{P}(u, v): u, v \in S, u \neq v\} \qquad (2)$$

and $V(\tilde{J}(S)) = \cup_{\tilde{p}\in\tilde{J}(S)} V(\tilde{P})$. If $V(\tilde{J}(S)) = V(G)$ for some $\tilde{J}(S)$ then the set S is called the Strong monophonic set. The minimum cardinality of the strong monophonic set is the strong monophonic number [10] . The monophonic distance $d_m(u, v)$ is defined as the length of a longest $u - v$ monophonic path in G. Many variants of the monophonic set that are equivalent to geodetic concepts had been studied in the literature. In this paper we define the Strong edge monophonic number of a graph and its properties.

Theorem 1 *[2] Each extreme vertex of a connected graph G belongs to every monophonic set of G. Moreover, if the set S of all extreme vertices of G is a monophonic set, then S is the unique minimum monophonic set of G.*

Theorem 2 *[8] Each simplicial vertex of G belongs to every edge monophonic set of G.*

Theorem 3 *[8] Let G be a connected graph with cut vertex V and let V be a edgemonophonic set of G. Then every component of $G - V$ contains an element of S.*

Theorem 4 *[8] No cut vertex of a connected graph G belongs to any minimum edge monophonic set of G.*

II. STRONG EDGE MONOPHONIC SET

The strong edge monophonic number of a graph $G = (V, E)$ is defined as follows. Let $S \subseteq V(G)$, then for each pair of vertices $u, v \in S$, $u \neq v$, let $\tilde{P}(u, v)$ be the selected fixed monophonic path between u and v . Then we set

$$\tilde{J}(S) = \{\tilde{P}(u, v) : u, v \in S, u \neq v\} \qquad (3)$$

and $E(\tilde{J}(S)) = \cup_{\tilde{P} \in \tilde{J}(S)} E(\tilde{P})$. If $E(\tilde{J}(S)) = E(G)$ for some $\tilde{J}(S)$ then the set S is called the Strong edge monophonic set. In other words a set $S \subseteq V(G)$ is called a strong edge monophonic set if each edge of G lies on one fixed monophonic path between pairs of vertices from S. The minimum cardinality of strong edge monophonic set is the strong edge monophonic number. It is denoted by $Sm_1(G)$.

Example 1: Consider the graph G given in fig1, the strong monophonic set is $S_1 = \{a, d, f\}$. But S_1 is not a strong edge monophonic set. Let $S_2 = \{a, b, c, d, e, d, f\}$ clearly S_2 will cover all the edges of the graph in a unique path. Hence $Sm_1(G) = 6$.

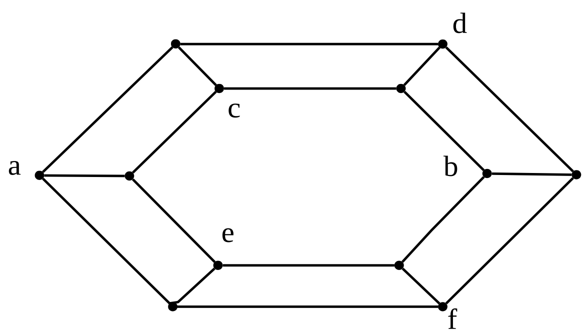

Figure 1:

A vertex v in a connected graph G is a simplicial vertex (or) extreme vertex if the subgraph induced by its neighbour is complete. Also a vertex v is said to be a semi simplicial vertex if$\Delta(< N(v) >) = |N(v) - 1|$. A set S of vertices of a connected graphG is called a cut set of G if the graph $G - S$ is connected. In particular, a vertex $v \in V(G)$ is a cut vertex of G if $G - V$ is disconnected.

Theorem 5 *If G is a graph with order n and monophonic diameter $d_m \geq 2$ then* $Sm_1 \geq \left\lceil \frac{1+\sqrt{1+8e/d_m}}{2} \right\rceil$.

Proof. Let $e = |E(G)|$, d_m the monophonic diameter and S be the minimum Strong edge geodetic set of G ,Such that $|S| = Sm_1(G)$. Since monophonic diameter of G is d_m, each monophonic path will have length at most d_m and covers at most edges of $G - S$. As the graph is covered with $\binom{Sm_1(G)}{2}$ monophonic paths,

$$e \leq \binom{|S|}{2} \, d_m$$

Which in turn implies

$$|S|^2 - |S| - \frac{2e}{d_m} \geq 0 \qquad (4)$$

Since Sm_1 is a non-negative integer we conclude

$$Sm_1 \geq \left\lceil \frac{1+\sqrt{1+8e/d_m}}{2} \right\rceil. \qquad (5)$$

Also the bound is sharp for a Path graph P_n and a Theta graph $\theta(5,3)$ as shown in the figures below.

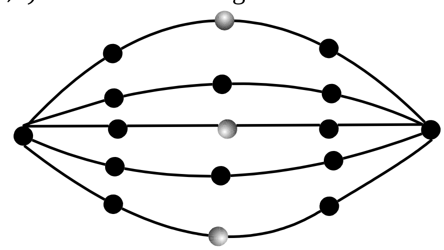

Figure 2:

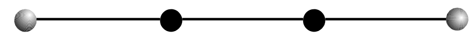

Figure 3:

Corollary 1 *For the complete graph $K_n (n \geq 2)$ $Sm_1(K_n) = n$.*

Proof. For a complete graph $K_n (n \geq 2)$ each vertex is a simplicial vertex. Also each strong edge monophonic set is a edge monophonic set, the result follows from theorem 2.

Theorem 6 *For a Connected graph G of order n, $2 \leq m(G) \leq m_1(G) \leq Sm_1(G) \leq n$.*

Proof. The set of two end vertices of a path P_n is its unique minimum monophonic set $m(G) \leq 2$. Also every strong edge monophonic set is a edge monophonic set and every edge monophonic set is a monophonic set , therefore $m(G) \leq m_1(G) \leq Sm_1(G)$. By Corollary 1, $Sm_1(K_n) \leq n$.

Remark *For a complete graph $Sm_1(G) = n$ and for a path graph P_n $Sm_1(G) = 2$. Hence the bounds in theorem 6 are sharp.*

For the graph G given in the fig 4, $m(G) = \{v_2, v_4\}$, $m(G) = 2$, $m_1(G) = \{v_1, v_6, v_7, v_3\}$, $m_1(G) = 4$ and $Sm_1(G) = \{v_1, v_6, v_7, v_3, v_2, v_4\}$, $Sm_1(G) = 6$. Therefore $2 \leq m(G) < m_1(G) < Sm_1(G) < n$.

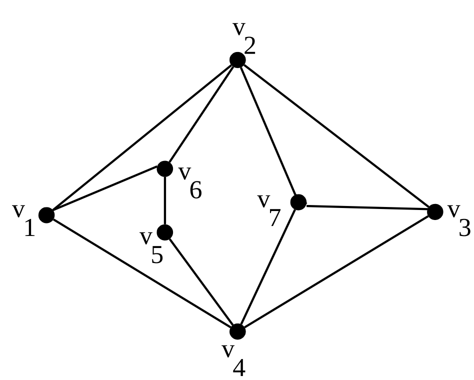

Figure 4:

Theorem 7 *Every Strong edge monophonic set of a graph contains its simplicial vertices.*

Proof: Since every strong edge monophonic set is a edge monophonic set the result follows from theorem 2.

Theorem 8 *Let G be a connected graph with order n and K simplicial vertices then $max\{2, K\} \le Sm_1(G) \le n$.*

Theorem 9 *For a $n-$ dimensional hexagonal silicate network $SL(n)$ with K simplicial vertices, $Sm_1(SL(n)) = K$.*

Proof: Let G be the $n-$ dimensional hexagonal silicate network. The simplicial vertices of $SL(N)$ are marked by white vertices in fig.5. It is easy to verify that the set of simplicial verices forms a strong monophonic set. Hence the result follows from theorem 8.

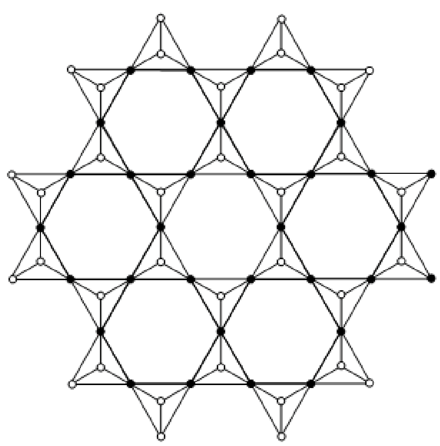

Figure 5:

Theorem 10 *Let G be a connected graph with cut vertex v. Then each Strong edge monophonic set contains at least one vertex from each component of $G - v$.*

Proof. The proof follows from theorem 3 as every strong edge monophonic set is a edge monophonic set.

Theorem 11 *For a non-trivial tree $T$with K extreme vertices, then the Strong edge monophonic number is the number of extreme vertices, $i.e)$ $Sm_1(T) = K$.*

Proof. Let S be the set of all end vertices of T. By theorem 7, $Sm_1(T) \ge S$. Now consider a vertex v such that v is a cut vertex of G. Since every strong edge monophonic set is a edge monophonic set and by theorem 4 it is clear that$v \notin S$. Also for an internal vertex v of T there exist end vertices x, y of T such that v lies on the unique $x - y$ monophonic path in T. Thus an internal vertex v of T , will lie on exactly $\binom{k}{2}$ distinct monophonic paths of vertices in S . Hence $Sm_1(T) = K$.

Corollary 2 *For positive integers k, n such that $2 \le k \le n$ there exist a connected graph$G$ of order n , with $m(G) = Sm_1(G) = k$.*

Proof. For $k = n$ the result follows from corollary1. Also for each pair of integers with $2 \le k \le n$, there exist a tree of order n and k extreme vertices. Hence the result follows from theorem 11.

Theorem 12 *Let G be a connected graph. Then $Sm_1(G) = 2$ if and only if $G \cong P_n$.*

Proof. Suppose $G \cong P_n$. Then it is obvious that $Sm_1(G) = 2$. Now let us assume that $Sm_1(G) = 2$. This is possible only if there exists two non-adjacent vertices u and v in the graph G, such that $E(\tilde{J}(u, v)) = E(G)$. In other words if there exists two non-adjacent vertices u and v such that all edges of G lie on a unique $u - v$ monophonic path. This implies that the graph G is a path graph.

Theorem 13 *If C_n is a cycle of order n, then $Sm_1(G) = 3$.*

Proof. By theorem 12 $Sm_1(G) > 2$. Any three vertices of a cycle will form a strong edge monophonic set. Hence $Sm_1(G) = 3$.

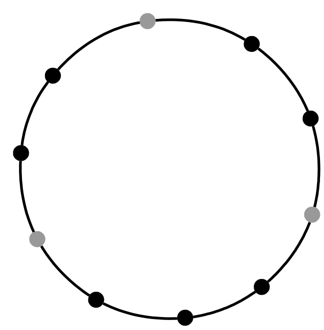

Figure 6:

Theorem 14 *For a theta Graph $\theta(l, k)$ the Strong edge monophonic number $Sm_1(\theta(l, k)) \le \left\lceil \frac{l}{2} \right\rceil$.*

Proof. Consider a theta graph $\theta(l, k)$. Let l be the number of levels of the theta graph and k be the number of vertices in each level.For a strong edge monophonic set , choose a vertex from alternate levels which forms a monophonic path as shown in figure 2. Hence $Sm_1(\theta(l, k)) \le \left\lceil \frac{l}{2} \right\rceil$.

Theorem 15 *For a Grid G_r, s the Strong edge monophonic number $Sm_1(G_r, s) = 4$.*

Proof. By theorem 12, $Sm_1(G_r, s) \ge 3$. Now assume that $Sm_1(G_r, s) = 3$. Let $S = (x_1, y_1), (x_1, y_s), (x_r, y_s)$ be the strong edge monophonic set of G_r, s then there exist three paths between the vertices of S which is not enough to cover all edges of G_r, s. For example consider the grid G_r, s in figure 7 . Hence $S = (x_1, y_1), (x_1, y_s), (x_r, y_s), (x_r, y_1)$, $Sm_1(G_r, s) = 4$ which is a contradiction to our assumption.

Also these $4c_2$ paths is enough to cover all the edges of the grid in a unique monophonic path. Therefore $Sm_1(G_r, s) = 4$

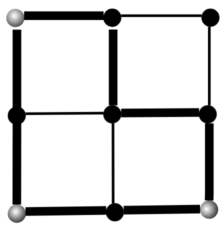

Figure 7:

Theorem 16 *For a Circulant graph $C(n, \{1,2\})$ The Strong edge monophonic number is n.*
Proof. Let G be a circulant graph $C(n, \{1,2\})$. Since we are considering a circulant graph $C(n, \{r, s\})$ with $\{r, s\} = \{1,2\}$, there exists a chord between any three vertices of the graph G. From the definition it clear that a monophonic path is a chord less path. Hence if $Sm_1(G) = v_1, v_2 \ldots . v_{n-1}$ leaving v_n, the edges $v_{n-1} - v_n$ and $v_n - v_1$ will not be covered. Therefore $Sm_1(C(n, \{1,2\})) = n$.

Corollary 3 If G is a connected graph of order $n \geq 3$ with exactly one universal vertex then the Strong edge monophonic number will be $Sm_1(G) = n - 1$.

Corollary 4 *For the Wheel graph and Star graph with* $(n \geq 3)$, $Sm_1(W_{1,n-1}) = Sm_1(K_{1,n-1}) = n - 1$.

Theorem 18 *For a non-trivial tree T of order n and monophonic diameter d_m , $Sm_1(T) = n - d_m + 1$ if an only if T is a Caterpillar.*
Proof. Let T be a non-trivial tree. Let $P: u = v_0, v_1, \ldots \ldots, v_{dm}$ be a monophonic diametrical path. Let l be the internal vertices of T other than $v_1, v_2 \ldots ., v_{dm-1}$ and K be the no. of end vertices of T. Then $n = d_m - 1 + l + k$. By theorem 11 , $Sm_1(T) = K$, and so $Sm_1(T) = n - d_m + 1 - l$. Therefore $Sm_1(T) = n - d_m + 1$ if and only if, $l = 0$, if an only if all the internal vertices of T lie on a monophonic diametrical path if and only if T is a caterpiller.

Theorem 19 For any graph G, $Sm_1(G) \leq Sg_e(G)$.

III. REALIZATION RESULT

Theorem 20 *For any two positive integers a, b with $a \geq b + 1$ and $b > 2$ there exist a connected graph G with $|V(G)| = a$.*
Proof. Let $P: u_0, u_1, \ldots . u_(a - (b + 1))$ be a path. Consider the graph G constructed from P by joining $b - 1$ new vertices. Then the graph$G$ is a tree with $|V(G)| = a - b + 1 + b - 1 = a$ and we have $b\ (b - 1 + 1)$number of leaves. Hence by theorem 11, $Sm - 1(G) = b$.

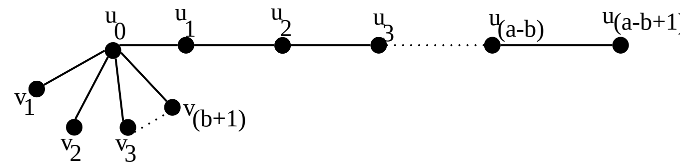

Figure 8:

Theorem 21 *For any positive integers $2 \leq a \leq b$ there exist a connected graphG such that $m(G) = a$ and $Sm - 1(G) = b$.*
Proof. If $a = b$, consider a tree with a end vertices. Then by theorem 11 $m(G) = Sm_1(G) = a$.

For $a = 2, b = 3$, then for the graph in the figure $m(g) = a$ and $Sm_1(G) = b$.

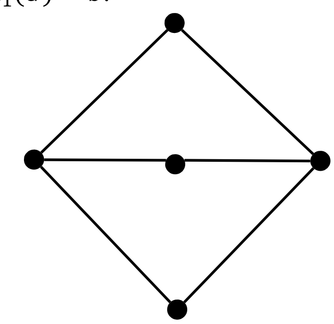

Figure 9:

If $a = 2\ \ b \geq 4$, consider the graph in figure, obtained from the path on five vertices $P: u_1, u_2, u_3, u_4, u_5$. Now add $b - 2$ independent vertices and join each $v_i (1 \leq i \leq b - 2)$ to u_2 and u_4. Then it is clear that the monophonic set $m(g) = u_1, u_5 = a$. From the definition of Strong edge monophonic set, there should be a unique monophonic path between any two vertices of the element of S. Hence $Sm_1(G) = \{u_1, u_5, v_1, \ldots . v_{b-2}\} = b$.

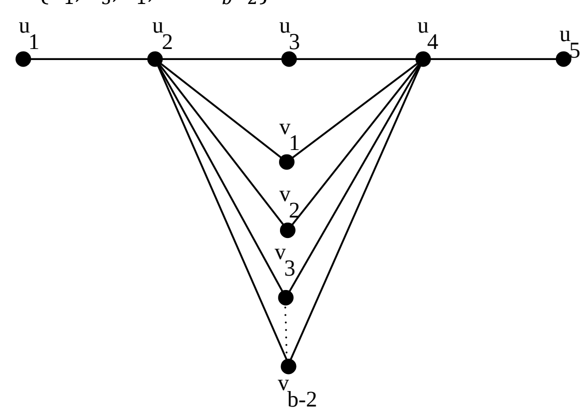

Figure 10:

If $a \geq 3$, $b \geq 4$, $b \neq a + 1$, the graph G is obtained from the path on five vertices $P: u_1, u_2, u_3, u_4, u_5$.. Now add $b - a$ independent vertices and join each $v_i (1 \leq i \leq b - a)$ to u_2 and u_4. Also add $\{w_1, w_2, \ldots\ldots w_{a-2}\}$ new vertices and join each $w_i (1 \leq i \leq a - 2)$ to u_2 and u_3. Since each w_i is a simplicial vertex of G, it is clear that it belongs to every monophonic set of G. Since $W = \{w_1, w_2, \ldots\ldots w_{a-2}\}$ is not a monophonic set of G, we have $W \cup \{u_1, u_5\}$ as the monophonic set of G. Therefore $m(G) = a$. Next we have to show $Sm_1(G) = b$. Obviously all the simplicial vertices belongs to the strong edge monophonic set. Since the strong edge monophonic set set should have a unique monophonic path any two vertices of S, $Sm_1(G) = \{w_1, w_2, \ldots\ldots w_{a-2}\} \cup \{i_1, u_5\} \cup \{v_1, v_2 \ldots\ldots, v_{b-a}\} = b$.

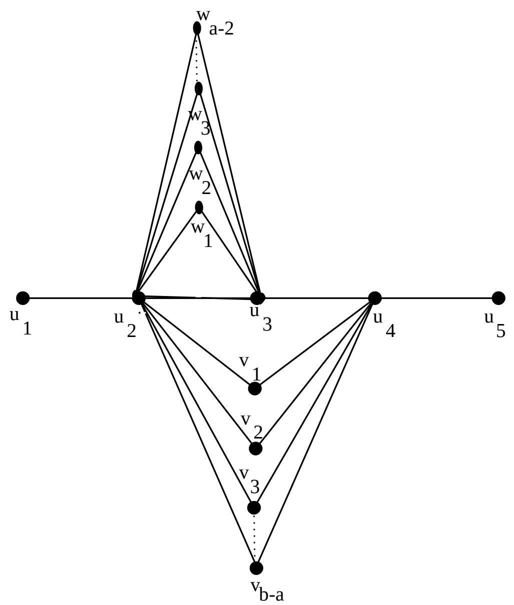

Figure 12:

For $a \geq 3$, $b \geq 4$ and $b = a + 1$, consider the graph G constructed as follows. Let $P: v_1, v_2, v_3, v_4$ and $P: u_1, u_2, u_3, u_4$ be two paths of length 4. let w and z be two independent vertices. Now join v_1 and u_1 to w ,and join v_4 and u_4 to z. Also add $x_i (1 \leq i \leq a - 1)$ independent vertices and add them to w. Now join z and u_3 . Let $S = \{x_1, x_2, \ldots . x_{a-1}, u_4\}$ be the set of simplicial vertices of G. It is clear that S is contained in every monophonic set of G. Hence $m(G) = a$. It is also clear that the set S is the strong monophonic set. But S is the not a strong edge monophonic set. However $S \cup \{z\}$ is a strong edge monophonic set. Hence $Sm_1(G) = b = a + 1$.

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