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Energy of Cartesian product of Graphs

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Abstract- An eigenvalue of a graph is an eigenvalue of its adjacency matrix. The energy of a graph is the sum of absolute values of its eigenvalues. Two graphs having same energy and same number of vertices are called *equienergetic graphs*. One might be interested to know, as to how the energy of a given graph can be related with the graph obtained from original graph by means of some graph operations. As an answer to this question we have considered the Cartesian product of two graphs. In this paper we obtain the eigenvalues and energy of Cartesian product of two graphs from the eigenvalue of the given graph.

***AMS Subject Classification*:** 05C50
Keywords-Cartesian Product, Adjacency Matrix, Eigenvalues, Energy of graph

I. INTRODUCTION

All graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [4] while for algebra we follow Lang [11].

The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, v_3, v_4, \dots \dots v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that, $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise.

The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots \dots . \lambda_n$ of the graph G are the eigenvalues of its adjacency matrix $[a_{ij}]$. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph and it is denoted by $Spec\ G$.

In 1978 Gutman [8] defined the energy of a graph G as the sum of absolute values of the eigenvalues of graph G and denoted it by $E(G)$. Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|$$

Here it has also been mentioned that the energy of totally disconnected graph ${K_n}^c$ is zero while the energy of complete graph K_n with maximum possible number of edges is $2(n-1)$. Therefore Gutman [7] leads to the conjecture that all graphs have energy at most $2(n-1)$. But then in [12] this was disproved since Hyperenergetic graphs are graphs for which the energy is greater than $2(n-1)$. The graph G is non-hyperenergetic if $E(G) \leq 2(n-1)$. In 2004, Bapat and Pati [5] proved that if the energy of a graph is rational then it must be an even integer, while Pirzada and Gutman [13] established that the energy of a graph is never the square root of an odd integer. A brief account of graph energy is given in [3] as well as in the books [5, 10]. Some fundamental results on graph energy are also reported in the thesis of Sriraj [15]. In 2011 Andries .E. Brouwer and Willem .H. Haemers [1] have found spectra of many graphs. In theoretical chemistry, using Huckel theory, the π-electron energy of a conjugated carbon molecule was computed, which coincides with the energy defined here. Hence results on graph energy assume special significance

The present work is intended to relate the graph energy to larger graphs obtained from the given graph by means of some graph operations. In [14] Samir K.. Vaidya and Kalpesh M. Popat have found the relation for energy of splitting graph and shadow graph given the energy of original graph by means of some graph operations. In this paper we have considered the Cartesian product of two graphs namely $K_2 \times G$ and obtained the energy from the eigenvalues of the given graph G.

Definition1.1. The Cartesian product of two simple graphs H and K is the graph $G = H \times K$ with $V(G) = V(H) \times V(K)$ in which vertices (h, k) and $(\acute{h}, \acute{k})$ are adjacent iff either

(1) $h = \acute{h}$ and $k, \acute{k}$ are adjacent in K, or
(2) $k = \acute{k}$ and $h, \acute{h}$ are adjacent in H.

II. Energy of $K_2 \times G$

The Cartesian product $K_2 \times G$ graph decomposes into a copies of G and b copies of K_2, where $n(K_2) = a$ and $n(G) = b$. By the definition of Cartesian product, $K_2 \times G$ has two types of edges: those whose vertices have the same first coordinate, and those whose vertices have the same second coordinate. The edges joining vertices with a given value of the first coordinate form a copy of G, so the edges of the first type form aG. Similarly, the edges of the second type form bK_2, and the union is $K_2 \times G$.

Theorem 2.1. If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots \ldots \lambda_n$ are the eigenvalues of G and $E(G) = \sum_{i=1}^{n} |\lambda_i|$, then

$$E(K_2 \times G) = \sum_{i=1}^{n} |\lambda_i(G) \pm 1|$$

Proof: Let $k_1, k_2, k_3, k_4, \ldots \ldots k_n$ be the vertices of the graph G. Then its adjacency matrix is given by

$$A(G) = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{array} \begin{array}{c} \begin{array}{ccccc} k_1 & k_2 & k_3 & \cdots & k_n \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Let h_1, h_2 be the vertices of the graph K_2. The vertex set
$V(K_2 \times G) = w_i =$
$\{(h_1, k_1), (h_1, k_2), \ldots (h_1, k_n), (h_2, k_1), \ldots (h_2, k_n)\}$
and $[(h_1, k_j), (h_2, k_l)] \in E(K_2 \times G)$ iff either

i) $h_1 = h_2$ & $k_j k_l$ are adjacent in G, or
ii) $k_j = k_l$ & $h_1 h_2$ are adjacent in K_2 .

Then, $A(K_2 \times G)$ can be written as a block matrix as given below,

$$\begin{array}{c|cccccccccc} & w_1 & w_2 & w_3 & \cdots & w_n & w_{n+1} & w_{n+2} & w_{n+3} & \cdots & w_n \\ \hline w_1 & 0 & a_{12} & a_{13} & \cdots & a_{1n} & 1 & 0 & 0 & \cdots & 0 \\ w_2 & a_{21} & 0 & a_{23} & \cdots & a_{2n} & 0 & 1 & 0 & \cdots & 0 \\ w_3 & a_{31} & a_{32} & 0 & \cdots & a_{3n} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ w_{n+1} & 1 & 0 & 0 & \cdots & 0 & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ w_{n+2} & 0 & 1 & 0 & \cdots & 0 & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ w_{n+3} & 0 & 0 & 1 & \cdots & 0 & a_{31} & a_{32} & 0 & \ddots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{2n} & 0 & 0 & 0 & \cdots & 1 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{array}$$

That is,

$$A(K_2 \times G) = \begin{bmatrix} A(G) & I \\ I & A(G) \end{bmatrix}$$

If A is an $n \times n$ matrix and suppose λ_A is an eigenvalue of A, with eigenvector $v_A \neq 0$, then we have,

$$Av_A = \lambda_A v_A$$

Let $B = \begin{bmatrix} A & I \\ I & A \end{bmatrix}$ with $2n$ eigenvectors $V_{A+} = \begin{pmatrix} v_A \\ v_A \end{pmatrix}$ and $V_{A-} = \begin{pmatrix} v_A \\ -v_A \end{pmatrix}$. By using laws of matrix algebra one can prove that $\lambda_A \pm 1$ are the eigenvalues of B for every eigenvalue λ_A of A. Thus every eigenvalue μ of B are precisely $\lambda_A \pm 1$, where λ_A ranges over the eigenvalues of A.

Hence, the eigenvalue of the above block matrix $\lambda_i(K_2 \times G)$ is calculated by adding one for every eigenvalue λ_i of G and subtracting one for every eigenvalue λ_i of G, (i.e)

$$E(K_2 \times G) = \sum_{i=1}^{n} |\lambda_i(K_2 \times G)|$$
$$= \sum_{i=1}^{n} |\lambda_i(G) \pm 1|$$

Illustration 2.1. Consider the cycle C_4 and the Cartesian product $K_2 \times C_4$. The energy $E(C_4) = 4$ as $Spec(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

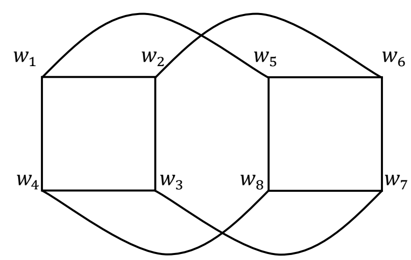

Figure 1: Cartesian product of $K_2 \times C_4$

$$A(K_2 \times G) = \begin{array}{c|cccc:cccc} & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\ \hline w_1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ w_2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ w_3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ w_4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline w_5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ w_6 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ w_7 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ w_8 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}$$

Therefore, $Spec(K_2 \times C_4) = \begin{pmatrix} 3 & -3 & -1 & 1 \\ 1 & 1 & 3 & 3 \end{pmatrix}$
Hence,

$$E(K_2 \times G) = 12$$

III. CONCLUSION

The energy of a graph is one of the emerging concepts within graph theory. This concept serves as a frontier between chemistry and mathematics. The energy of many graphs is known. But we have take up the problem to investigate the energy of $K_2 \times G$ given the energy of G.

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