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Stability of Fractional Order System of Duffing Equation with Quadratic and Cubic Nonlinearities

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Abstract – In physics, mechanics and engineering, Duffing equations are used in describing the oscillatory systems with non linearities and is famous in study of nonlinear dynamics. Here, we study the asymptotic stability of the fractional order unforced damped Duffing equation with quadratic nonlinearity. Local asymptotic stability conditions for commensurate order fractional derivative system with order lying in (0,2) is discussed without considering integer order. The stability of the system is investigated with fractional orders in two ranges (0,1) and (1,2). For different values of the parameters, examples with simulations are performed. Sensitivity of the system for the small variation in fractional order is analyzed with 2-Dimensional time plots. Lyapunov exponents for the system is investigated with plots and values of Lyapunov exponents are tabulated.

Keywords - Duffing equation, Fractional order system, Stability Nonlinear system

I. INTRODUCTION

It was late 17^{th} century which saw the origin of both Classical and Fractional calculus. The solving methodology of the fractional order differential equations was developed in the 20^{th} century, that is why the use of integer order derivatives in modeling of real life systems become popular. Fractional Calculus gives better characterization of many physical phenomena. Advantages in modeling a real life process with fractional derivatives is its relation to whole space and whole time domain. The dynamical behavior exhibited by the physical system due to the presence of some special materials and chemical properties are more precisely described by Fractional derivatives [[2],[5]].

Importance of stability of non-linear fractional order systems in control theory has attracted many researchers [8]. In the study of stability of fractional derivative systems, commensurability of the system plays a vital role. [[9],[4]] In non linear dynamics, mostly studied oscillators behavior are represented by Van der Pol equation and Duffing equation. Dynamic analysis of autonomous and forced cases of oscillators is carried out with different variations in the Duffing equations [[7],[3]]. In this paper, we study the stability of commensurate non-integer order system of unforced Duffing oscillator with quadratic term and order lying in (0,2).

This work is organized as follows. In Section 2, some necessary theorems are provided. The stability of Duffing oscillator system with order in (0,1) and (1,2) are discussed with numerical simulations in Section 3. Lyapunov exponent of the system are presented in Section 4 and conclusion in Section 5.

II. PRELIMINARIES

The following theorems are useful in the discussion of the stability of commensurate non integer order models. Consider the general form of nonlinear fractional derivative system with order ν

$$\frac{d^{\nu}w(t)}{dt^{\nu}} = \Psi(w(t)) = Bw(t) + \Phi(w(t)) \qquad (1)$$

where state vectors are denoted by $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T \in \Re^n$, $\Psi: \Re^n \to \Re^n$ is non linear mapping. $B \in \Re^{n\times n}$, $\Psi(w(t))$ itself contains the linear part $Bw(t)$ and nonlinear part $\Phi(w(t))$.

Theorem 2.1 *[1] The system (1) with order $0 < \nu < 1$ is locally asymptotically stable, if*

1. $\Phi(w(t))$ satisfies $\Phi(0) = 0$ and $\lim_{x\to 0} \frac{\|\Phi(w(t))\|}{\|w(t)\|} = 0$.
2. B is a Hurwitz matrix.

Theorem 2.2 *[6] The system (1) with order $1 < \nu < 2$ is locally asymptotically stable, if*

1. $\Phi(w(t))$ satisfies $\Phi(0) = 0$ and $\lim_{w\to 0} \frac{\|\Phi(w(t))\|}{\|w(t)\|} = 0$.
2. $Re\lambda(B) < 0$ and $\zeta = -maxRe\lambda(B) > (\Gamma(\nu))^{\frac{1}{\nu}}$.

III. MATHEMATICAL MODEL AND STABILITY ANALYSIS

The 2-Dimensional system representing the unforced Duffing oscillator with quadratic term is expressed as follows.

$$\frac{d^{\nu}p(t)}{dt^{\nu}} = u(t);$$
$$\frac{d^{\nu}u(t)}{dt^{\nu}} = -\mu u(t) - \kappa p(t) - (\eta + \rho p(t))(p(t))^2; \qquad (2)$$

where $u(t)$ represents the fractional damping of the system with ν being the fractional order. μ, κ, η and ρ are constants and assumed to take positive values.

The alternate form of the system (2) in terms of (1) is

$$\frac{d^{\nu}w(t)}{dt^{\nu}} = \begin{bmatrix} 0 & 1 \\ -\kappa & -\mu \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ -(\eta + \rho p(t))(p(t))^2 \end{bmatrix} \quad (3)$$

where, $w(t) = [p(t) \quad u(t)]^T$.

A. Stability Analysis of System (3)

The basic requirement for system analysis and system synthesis is the study of stability. The stability of system is discussed neglecting the integer order in the range $(0,2)$.

1) Fractional order $\mathbf{0 < \nu < 1}$

For the system (3) with order $\nu \in (0,1)$ stability conditions are obtained using Theorem (1).

Condition (i): If $w(t) = [p(t) \quad u(t)]^T = 0$, then clearly $\lim_{w \to 0} \frac{\|\Phi(w(t))\|}{\|w(t)\|} = 0$.

Condition (ii): To show that the matrix B is a Hurwitz matrix, it is enough to prove that every eigen values of the matrix B has strictly negative real parts.

$$B = \begin{bmatrix} 0 & 1 \\ -\kappa & -\mu \end{bmatrix}$$

Eigenvalues of B are $\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4\kappa}}{2}$. There arises three cases in order the eigen values to have negative real parts.

- $\mu^2 < 4\kappa$
- $\mu^2 = 4\kappa$
- $\mu^2 > 4\kappa$

But clearly if $\mu^2 \leq 4\kappa$ there is no effect on the real part of the Eigenvalues. In case of $\mu^2 > 4\kappa$, the value of $\sqrt{\mu^2 - 4\kappa}$ is strictly less than μ as $\kappa > 0$.

2) Numerical Examples

Example 1. *For Commensurate order* $\nu = 0.95$ *and the parameter values* $\mu = 0.25, \kappa = 1, \eta = 0.6, \rho = 1.2$ *and the initial condition is (0.02,0.03), the system (3) is given by*

$$\frac{d^{0.95}}{dt^{0.95}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.25 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(0.6 + 1.2\ p(t))(p(t))^2 \end{bmatrix}$$

The eigen value of the linear term is given by
$\lambda_{1,2} = -0.125 \pm 0.99215i$ clearly $Re(\lambda_{1,2}) < 0$.

By Theorem (1) it is clear that the system (3) is Locally asymptotically stable. Under damped motion of the system is clearly visible from the time plots and inward spiral motion in Phase plane diagram explains stability as in Figure (1).

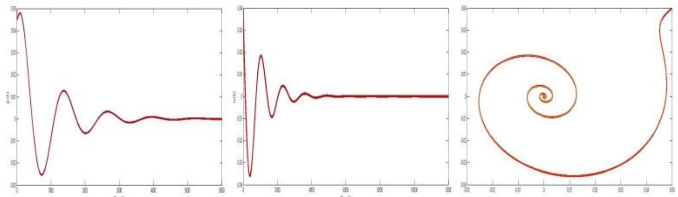

Figure 1: Stability of the system for case (i)

Example 2. *The system*

$$\frac{d^{0.95}}{dt^{0.95}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.25 & -1 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(0.6 + 1.2\ p(t))(p(t))^2 \end{bmatrix}$$

with initial condition (0.2,0.3) has the eigen values $\lambda_{1,2} = -0.5 < 0$. Thus, the system (3) is Locally asymptotically stable. From Figure (2), critical damping motions and motion of the system towards fixed point is illustrated.

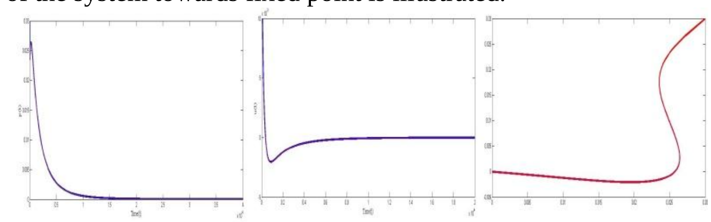

Figure 2: Stability of the system for case (ii)

Example 3. *The system*

$$\frac{d^{0.90}}{dt^{0.90}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.1 & -1 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(0.8 + 0.9\ p(t))(p(t))^2 \end{bmatrix}$$

with initial condition (0.05,0.01) has distinct real eigen values $\lambda_1 = -0.11270 < 0$ and $\lambda_2 = -0.88729 < 0$. Thus,the system (3) is Locally asymptotically stable. The time plots in Figure (3) explains the over damping motions. Phase portrait explains the motion of the system towards origin thus confirming stability.

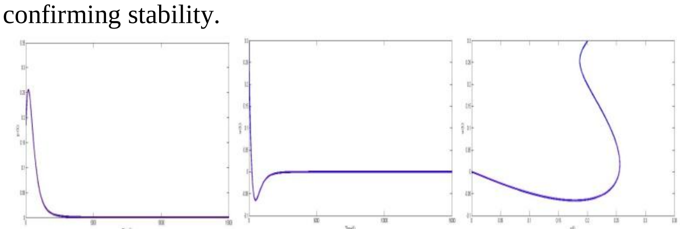

Figure 3: Stability of the system for case (iii)

B. Fractional order $\mathbf{1 < \nu < 2}$

The order of fractional derivative belonging to (0,1) is main focus on stability analysis of fractional derivative systems. The fractional order system with order lying in (1,2) also describes real life models like super- diffusion. Thus, we discuss the stability of system with order $1 < \nu < 2$.

The Stability conditions for the system (3) with $1 < \nu < 2$ is obtained from Theorem (2) as follows,
Condition (i) of the theorem is satisfied by the non linear term of the system (3) when $\Phi(0) = 0$.
Condition (ii) The real part of the eigenvalues are strictly negative which for the system (3) is clear from the above section. Our aim is to now check

$$\zeta = -maxRe\lambda(B) > (\Gamma(\nu))^{\frac{1}{\nu}} \quad (4)$$

The eigen values of the matrix

$$B = \begin{bmatrix} 0 & 1 \\ -\kappa & -\mu \end{bmatrix}$$

from System (3) are $\lambda_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4\kappa}}{2}$. We know that the $\Gamma(\nu) < 1, \forall \nu \in (1,2)$. Thus, the system (3) is locally asymptotically stable if maximum of the real parts of the eigenvalues are greater than 1. Thus, $-maxRe\lambda(B) > 1$.

1) Numerical examples

Example 4. *Taking* $\nu = 1.2$ *and* $\mu = 1.9, \kappa = 3, \eta = 1.1, \rho = 2$ *and the initial condition is (0.5,0.4), the system (3) becomes*

$$\frac{d^{1.2}}{dt^{1.2}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.9 & -3 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(1.1 + 2\ p(t))(p(t))^2 \end{bmatrix}$$

The eigen value of matrix B is given by $\lambda_{1,2} = -0.95 \pm 1.44827\ i$ clearly $Re(\lambda_{1,2}) < 0$. From (4), $\zeta = 0.95 > 0.9313267701$.By Theorem (2) the system (3) is Locally asymptotically stable. Time plots and phase portrait of the system for the corresponding parametric values are given in Figure (4).

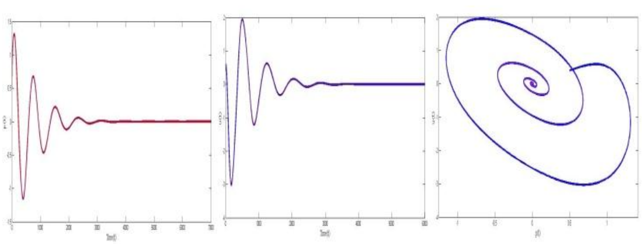

Figure 4: Stability of the system for $1 < \nu < 2$

Example 5. *The system*

$$\frac{d^{1.6}}{dt^{1.6}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(1 + 1.6\ p(t))(p(t))^2 \end{bmatrix}$$

with initial condition (0.005,0.04) has the eigen values $\lambda_{1,2} = -2 < 0$. $\zeta = 2 > 0.93204$ is obtained from (4). Time plot and Phase portrait in the Figure (5) explains local asymptotic stability of the system (3) .

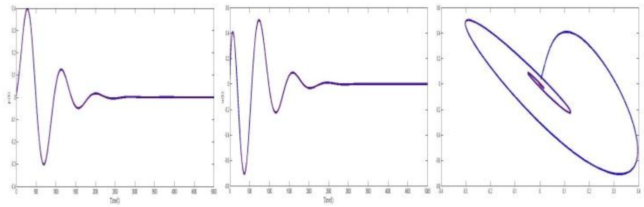

Figure 5: Stability of the system for $1 < \nu < 2$

Example 6. *The system*

$$\frac{d^{1.5}}{dt^{1.5}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4.5 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(1 + 1.6\ p(t))(p(t))^2 \end{bmatrix}$$

with initial condition (0.005,0.04) has distinct real eigen values $\lambda_1 = -1.21922 < 0$ and $\lambda_2 = -3.28077 < 0$. Using (4), we have $\zeta = -1.21922 > 0.92263$ which confirms that the system (3) is Locally asymptotically stable. Timeplots and Phase portrait are provided to strengthen our results in Figure (6).

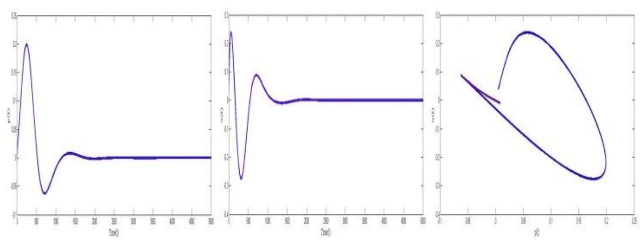

Figure 6: Stability of the system for $1 < \nu < 2$

Example 7. *Considering the system*

$$\frac{d^{1.14}}{dt^{1.14}} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -0.6 \end{bmatrix} \begin{bmatrix} p(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -(1.1 + 2\ p(t))(p(t))^2 \end{bmatrix}$$

with initial condition (0.01,0.01) has complex eigen values $\lambda_{1,2} = -0.3 + 1.38202\ i$. The real part of the eigen value is less than zero.But it fails to satisfy the (4) i.e., $\zeta = 0.3 < 0.94400$. Thus by Theorem (2) the system is not stable which is clearly confirmed by diverging oscillations in time plots and Limit cycle formed as in Figure (7).

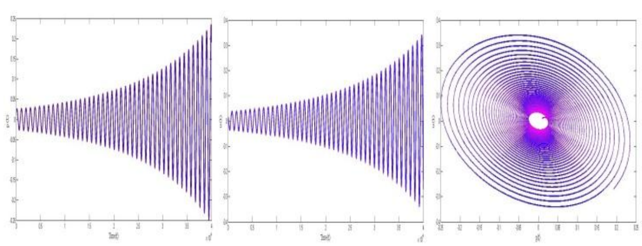

Figure 7: Limit Cycle of the system for $1 < \nu < 2$

IV. LYAPUNOV EXPONENTS OF THE SYSTEM (3)

Lyapunov exponent, a quantity in dynamical system, gives the characterization separation rate of nearby trajectories. Chaotic nature of the system is described by the positive lyapunov exponents. The Lyapunov exponents of the system (3) is determined by using Benettin-Wolf algorithm.

The Lyapunov exponent for the commensurate system (3) with order $\nu = 0.999$ and initial condition $(0.1, 0.1)$ and the values of parameters as $\mu = 0.005, \kappa = 2, \eta = 1.1, \rho = 2$ is calculated. The evolution time of the Lyapunov exponents are given in the Table (1). The graph in Figure (8) is plotted with Lyapunov exponents against time for the values in Table (1).

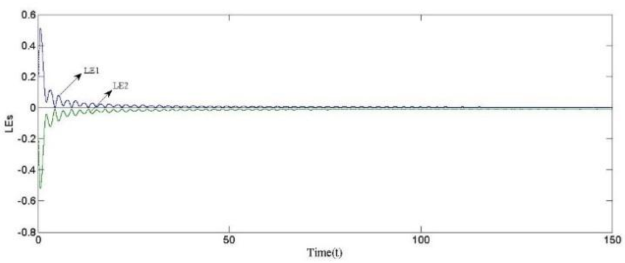

Figure 8: Lyapunov Exponent of the system (3)

Table 1: Time Evolution of LE

Time	LE1	LE2	Time	LE1	LE2
5	0.0654	-0.0704	80	0.0072	-0.0122
10	0.0419	-0.0469	85	0.0051	-0.0101
15	0.0117	-0.0167	90	0.0025	-0.0075
20	0.0093	-0.0143	95	0.0044	-0.0094
25	0.0163	-0.0213	100	0.0053	-0.0103
30	0.0123	-0.0173	105	0.0033	-0.0083
35	0.0030	-0.0080	110	0.0021	-0.0071
40	0.0090	-0.0140	115	0.0047	-0.0098
45	0.0116	-0.0166	120	0.0045	-0.0095
50	0.0060	-0.0110	125	0.0021	-0.0072
55	0.0039	-0.0089	130	0.0026	-0.0076
60	0.0074	-0.0124	135	0.0039	-0.0089
65	0.0072	-0.0122	140	0.0032	-0.0082
70	0.0031	-0.0081	145	0.0014	-0.0064
75	0.0042	-0.0092	150	0.0029	-0.0079

V. CONCLUSION

Local asymptotic stability conditions for the commensurate fractional derivative system with order in (0,2) is discussed in this paper in two parts one with order in range (0,1) and other in range (1,2).Numerical examples for different parameteric values and simulations are presented. Lyapunov exponents values are tabulated and plotted against time. Thus, the complex dynamics exhibited by the system is studied.

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