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Circular Geodetic Number of Certain Graphs

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***Abstract*—**A new variant geodetic problem, circular geodetic is defined as follows: Let $\boldsymbol{S} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_k}, \boldsymbol{x_{k+1}} = \boldsymbol{x_1}\}$ be a geodetic set of $\boldsymbol{G}$. Then $\boldsymbol{S}$ is said to be a circular geodetic set of $\boldsymbol{G}$, there exists an index $\boldsymbol{i}$, $\boldsymbol{1 \leq i \leq k}$, such that $\boldsymbol{I[x_i, x_{i+1}]}$ contains atleast a vertex $\boldsymbol{v}$ other than $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$, also $\boldsymbol{I[S] = V(G)}$. The minimum number of vertices needed to form a circular geodetic set is called circular geodetic number of $\boldsymbol{G}$ and it is denoted by $\boldsymbol{g_{cir}(G)}$.

***Keywords*—** Circular geodetic, Complete bipartite, Hexagonal mesh network, Apollonian.

I. INTRODUCTION

We consider a finite graph without loops and multiple edges. Let G be a graph the vertex set $v(G)$ and the edge set $E(G)$.The *order* of the graph G is $|V(G)|$ and *size* is $|E(G)|$. The *degree* $d(v)$ of the vertex $v\epsilon V(G)$ is the number of the edges adjacent to v i.e., $d(v) = |N(v)|$. For a vertex $v \epsilon V(G)$, the open neighborhood $N(v)$ is the set of all vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote the maximum and minimum degree of the graph G respectively.

If G is a graph then its complement is denoted by $\overline{G}$. The *girth* of a graph G is the length of the shortest cycle. The *Triangle free graph* is an undirected graph in which no three vertices form a triangle of edges.In a graph G a vertex x is simplicial if its neighborhood $N(x)$ induces a complete subgraph of G. If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest x -y path in G. The *diameter* of a connected graph G is defined by $diam(G) = max_{x,y\in V(G)}\, d(x, y)$. A $x - y$ path of length $d(x, y)$ is called a $x - y$ *geodesic.* The closed interval $I[x, y]$ is the set of vertices of all $x - y$ geodesic of G. For $S \subseteq V(G)$, $I[S] = \bigcup_{x,y\in S} I[x, y]$. A set S of vertices of a graph G is a geodetic set if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$.

Let $S = \{x_1, x_2, \ldots, x_k, x_{k+1} = x_1\}$ be a geodetic set of G. Then S is said to be a circular geodetic set of G, there exists an index i, $1 \leq i \leq k$, such that $I[x_i, x_{i+1}]$ contains atleast a vertex v other than x_i and x_{i+1}, also $I[S] = V(G)$. The minimum number of vertices needed to form a circular geodetic set is called circular geodetic number of G and it is denoted by $g_{cir}(G)$.

In this paper, the circular geodetic number of certain graphs and networks are presented and realizzation results are also given.

II. PRELIMINARIES

Definition 2.1 *A vertex $u \in V(G)$ is said to be geodominated by the pair $\{x, y\}$ if u lies on some $x - y$ geodesic in G, for any $x, y \in V(G)$. The geodetic interval $I[x, y]$ consists of x, y together with all vertices geodominated by the pair $\{x, y\}$. If S is a set of vertices of G, then the geodetic closure $I[S]$ is the union of all sets $I[x, y]$ for $x, y \in S$. If $I[S] = V(G)$, then S is said to be a geodetic set of G. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set.*

Definition 2.2 Let $S = \{x_1, x_2, \ldots, x_k, x_{k+1} = x_1\}$ be a geodetic set of G. Then S is said to be a circular geodetic set of G, there exists an index i, $1 \leq i \leq k$, such that $I[x_i, x_{i+1}]$ contains atleast a vertex v other than x_i and x_{i+1}, also $I[S] = V(G)$. The minimum number of vertices needed to form a circular geodetic set is called circular geodetic number of G and it is denoted by $g_{cir}(G)$.

Definition 2.3 *A vertex v of a graph G is called simplicial if its neighborhood $N(v)$ induces a clique*

Example 2.1 In Fig. 1, there are 3 simplicial vertices, $S = \{u_1, u_3, u_5\}$ and these vertices forms a geodetic set as well as circular geodetic set. But this fails to make the linear geodetic set. When we add the vertex u_6 to S, this becomes the linear geodetic set. This is the example for $g_{cir}(G) < g_l(G)$.

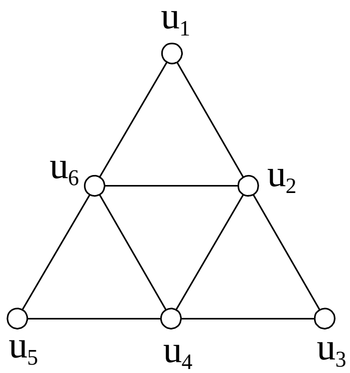

Figure 1

III. MAIN RESULTS

Proposition 3.1 *Every circular geodetic set of a graph contains all its extreme vertices. In particular, if the set of simplicial vertices S of G is a circular geodetic set of G, then S is the unique minimum circular geodetic set of G.*

Theorem 3.2 *Let G be any graph, then* $2 \leq g(G) \leq g_{cir}(G) \leq n$

Proof. It is clear that, $2 \leq g(G) \leq n$. By the definition 2.2, it is obvious that $g(G) \leq g_{cir}(G) \leq n$. Therefore we get $2 \leq g(G) \leq g_{cir}(G) \leq n$.

Theorem 3.3 *If G is a nontrivial connected graph of order n and diameter d, then $g_{cir}(G) \leq n - d + 1$.*

Theorem 3.4 *Let G be a connected graph. $g(G) = 2$ iff* $g_{cir}(G) = 2$.

Proof. In a connected graph G with $g(G) = 2$, we denote the geodetic set $S = \{a, b\}$. This shows that the vertices of $V \setminus S$ lies on the geodesics between a and b. It is clear that, $g_{cir}(G) = 2$. Now, let $g_{cir}(G) = 2$ and $S_{cir} = \{a, b\}$. This implies that the vertices of $V \setminus S_{cir}$ lies on the geodesics between a and b. And it is enough to show that $g(G) = 2$.

Theorem 3.5*For the complete bipartite graph $K_{m,n}$, $(2 \leq m \leq n)$*

$$g_{cir}(K_{m,n}) = \begin{cases} 2, & if m = 2 \\ 3, & if m = 3 \\ 4, & otherwise \end{cases}$$

Proof. For $(2 \leq m \leq n)$. Let X and Y be the partition of the complete bipartite graph, $K_{m,n}$.

Case (i): When $m = 2$.
Clearly, X is a circular geodetic set of $K_{2,n}$. Therefore $g_{cir}(K_{2,n}) = 2$.

Case (ii): When $m = 3$.
It is clear that, no two vertex subset is a circular geodetic set of $K_{3,n}$. Therefore $g_{cir}(K_{3,n}) = 3$.

Case (iii): When $m \geq 4$.
No three vertex subset will form a circular geodetic of $K_{m,n}$. Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be the partite sets of $K_{m,n}$. Then $S = \{x_1, x_2, y_1, y_2\}$ is a circular geodetic set of $K_{m,n}$. Therefore $g_{cir}(K_{m,n}) = 4$.

Theorem 3.6*For the cycle C_n,*

$$g_{cir}(C_n) = \begin{cases} 2, & ifniseven \\ 3, & ifnisodd \end{cases}$$

Proof. Case (i): When n is even.
Any two antipodal vertices of C_n are enough to form a circular geodetic set. Then $g_{\{}cir\} = 2$.

Case (ii): When n is odd.
Let $S = \{x_1, x_2, x_3\}$ be the set of vertices such that x_1 and x_2 are antipodal to x_3. Therefore S is a circular geodetic and so $g_{cir}(C_n) = 3$

Theorem 3.7*For a nontrivial tree T with k end vertices,* $g_{cir}(T) = k$.

Proof. Let T be a tree with k end vertices and $V(T) = n$. Clearly $g(T) = k$ and this implies $g_{cir}(T) \geq k$. In any arbitrary tree which contains k end vertices, we sequentially arrange the end vertices and label from 1 to k . Clearly $g_{cir}(T) \leq k$. Therefore $g_{cir}(T) = k$

Theorem 3.8*For a Hexagonal Mesh* Network, $g_{cir}(HX) = 3$.

Proof. Let $S = \{v_1, v_2, v_3\}$ as shown in Fig. 2. Without loss of generality assume that $S \setminus v_3$ be the geodetic set. $I[v_1, v_2]$ is not enough to cover all the vertices of G . Therefore $g_{cir}(HX) \geq 3$. And S clearly geodominates all the vertices of G. Hence $g_{cir}(HX) \leq 3$. Therefore $g_{cir}(HX) = 3$.

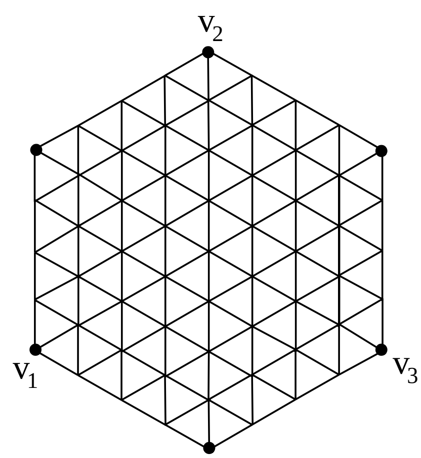

Figure 2

Theorem 3.9 *For a Hexagonal Mesh Pyramid* Network, $g_{cir}(HXP_n) = 3$.

Proof. Let $S = \{v_1, v_2, v_3\}$ as shown in Fig. 3. Without loss of generality assume that $S \setminus v_3$ be the geodetic set. $I[v_1, v_2]$ is not enough to cover all the vertices of G. Therefore $g_{cir}(HXP_n) \geq 3$. And S clearly geodominates all the vertices of G. Hence $g_{cir}(HXP_n) \leq 3$. Therefore $g_{cir}(HXP_n) = 3$.

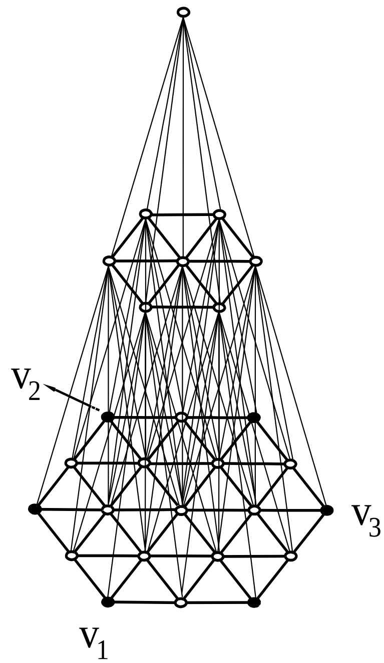

Figure 3

Theorem 3.10 *Let $A(r)$ be the complete Apollonian network of level r. Then $g_{cir}(A(r))$ is the set of all simplicial vertices.*

Proof. Let the simplicial vertices denote the set S. Then it is clear that $g_{cir}(A(r)) \geq |S|$. The complete Apollonian network of level r has 3^{r-1} simplicial vertices, for $r \geq 2$, whereas $A(0)$ and $A(1)$ has 3 and 4 simplicial vertices respectively. Since S construct the geodetic set for $A(r)$. Then $g_{cir}(A(r)) \leq |S|$. Therefore $g_{cir}(A(r)) = |S|$.

IV. REALIZATION RESULTS

Theorem 4.1 *If n, d and k are integers such that $2 \leq d < n$, $2 \leq k \leq n - d + 1$, then there exists a graph G of order n, diameter d and $g_{cir}(G) = k$.*

Proof. Let $P_d: u_0, u_1, \ldots, u_d$ be a path of length d. We first add $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ and join each to u_1 producing a tree T. Then we add new vertices $w_1, w_2, \ldots, w_{n-d-k+1}$ and join each to u_0, u_1 and u_2, thereby producing the graph G. Then G has order n and diameter d. Let $ud = v_{k-1}$ and $u_o = v_k$. First, let $d \geq 3$. Then it is clear that $S = \{v_1, v_2, \ldots, v_{k-2}, v_{k-1}\}$, the set of end vertices of G is not a circular geodetic set of G and so $g_{cir}(G) > k - 1$. On the other hand, $S_1 = \{v_1, v_2, \ldots, v_{k-1}, v_k\}$ is a circular geodetic set of G and so by Proposition 3.1, $g_{cir}(G) = k$. Next, let $d = 2$. Then, as above, it is easily seen that $S = v_1, v_2, \ldots, v_{k-2}, v_{k-1} = u_2, v_k = u_0$ is a minimum circular geodetic set of G and so $g_{cir}(G) = k$.

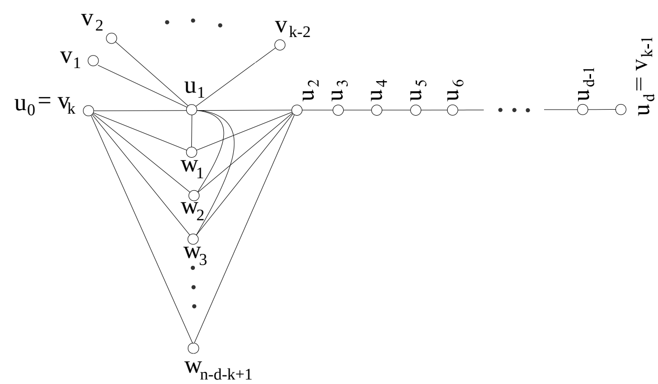

Figure 5

V. CONCLUSION AND FUTURE SCOPE

Circular geodetic number for tree, complete graph, complete bipartite graph, hexagonal mesh network, hexagonal mesh pyramid network, apollonian network are computed. And realization results are obtained.

REFERENCES

[1] Jurczyk, M., Siegel, H. J., Stunkel, C. B., Interconnection Networks for Parallel Computers, Wiley Encyclopedia of Computer Science and Engineering, 2008.

[2] G.Chartrand, F. Harary, H. C. Swart and P. Zhang, Geodomination in Graphs, Bulletin of the ICA, 31(2001), 51-59.

[3] Pelayo, Ignacio M. Geodesic convexity in graphs. New York: Springer, 2013.

[4] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Can. J. Math. **9** (1957) 515-525.

[5] Pelayo, Ignacio M. *Geodesic convexity in graphs*. New York: Springer, 2013.

[6] Hansberg, Adriana, and Lutz Volkmann. "*On the geodetic and geodetic domination numbers of a graph*." *Discrete Mathematics* 310.15 (2010): 2140-2146.

