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Induced P_3-Packing k-partition Number for Certain Graphs

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Abstract: Finding a partition $\boldsymbol{V_1, V_2, \ldots, V_k}$ of $\boldsymbol{V(G)}$ with minimum $\boldsymbol{k}$ is called the induced $\boldsymbol{H}$-packing $\boldsymbol{k}$-partition problem of $\boldsymbol{G}$. The minimum induced $\boldsymbol{H}$-packing $\boldsymbol{k}$-partition number is denoted by $\boldsymbol{ipp(G, H)}$. In this paper we determine an induced $\boldsymbol{P_3}$-packing $\boldsymbol{k}$-partition number for Butterfly Networks, Honeycomb Networks, and Circum Pyrene with $\boldsymbol{H}$ is isomorphic to $\boldsymbol{P_3}$.

Keywords - Perfect $\boldsymbol{P_3}$ −packing, Almost Perfect $\boldsymbol{P_3}$-packing, Induced $\boldsymbol{H}$-packing $\boldsymbol{k}$-partition, Butterfly networks, Honeycomb Networks and Circum Pyrene.

I. INTRODUCTION

Mathematically, assembling in predictable patterns is equivalent to packing in graphs. An H - packing of a graph G is the set of vertex disjoint subgraphs of G, each of which is isomorphic to a fixed graph H[1] . The maximum number of vertex disjoint copies of H in G is called the packing number and is denoted by $\lambda(G, H)$. An H-packing in G is called perfect if it covers all the vertices of G. Packing problems was already studied for Honeycomb Networks[10]. An H-packing is of practical interest in the areas of scheduling [2], wireless sensor tracking[3], wiring-board design, code optimization[5], and many others. Packing lines in a hypercube has been studied in[4]. H-packing is determined for honeycomb[10] and hexagonal network[6]. Partitioning a network with respect to vertices, edges or subgraphs is a significant aspect in enlarging resource utilization of parallel machines. Partitioning large networks is often important for complexity reduction or parallelization. For instance, in telecommunication networks, same frequency can be assigned to different sub networks if the frequencies do not interfere with each other. Thus the study of partitioning a H-packing such that no two members in the same partition interfere, becomes meaningful.

The concept is as follows: A collection $\mathcal{K} = \{H_1, H_2, \ldots, H_r\}$ of induced subgraphs of a graph G is said to be sg-*independent* if (i) $V(H_i) \cap V(H_j) = \phi$, $i \neq j$, $1 \leq i, j \leq r$. (ii) no edge of G has its one end in H_i and the other end in H_j, $i \neq j$, $1 \leq i, j \leq r$. If $H_i \simeq H$, $\forall i$, $1 \leq i \leq r$, then $\mathcal{K}$ is referred to as a H-independent set of G. Let $\mathcal{H}$ be a perfect or almost perfect H-packing of a graph G. Finding a partition $\{\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_k\}$ of $\mathcal{H}$ such that $\mathcal{H}_i$ is H-independent set, $\forall i$, $1 \leq i \leq k$, with minimum k is called the induced H-packing k-partition problem of G. The minimum induced H-packing k-partition number is denoted by $ipp_{\mathcal{H}}(G, H)$ and is defined as $ipp(G, H) = \min ipp_{\mathcal{H}}(G, H)$ where the minimum is taken over all H-packing of G. The induced H-packing k-partition problem was studied for certain interconnection networks such as hypercubes, Sierpiński graphs [9]. Xavier et al [9] proved that the induced P_3-packing k-partition problem is NP-complete, also induced C_4-packing k-partition problem is NP-complete.

In this paper we determine an induced H-packing k-partition number for Butterfly networks, Honeycomb Networks and Circum Pyrene with H is isomorphic to P_3.

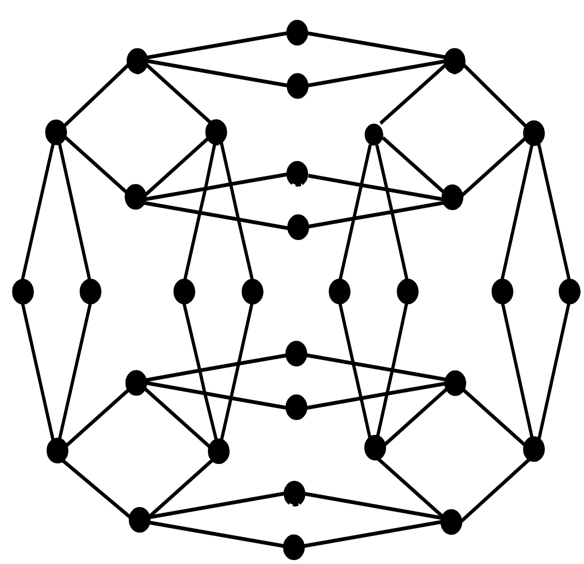


Figure 1: Structure of the Diamond representation of BF(3)

II. BUTTERFLY NETWORK

Efficient representation for butterfly and benes networks has been obtained by Manuel et al. [7]. The definition of the butterfly network BF(r) is as follows: The r-dimensional butterfly has $(r+1)2^r$ nodes and $r2^{r+1}$ edges. The set V of nodes of an r-dimensional butterfly correspond to pairs $[w,i]$, where i is the dimension or level of a node ($0 \le i \le r$) and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if = $i' = i + 1$ and either:

1. w and w' are identical, or
2. w and w' differ in precisely the i^{th} bit. The edges in the network are undirected. An r-dimensional butterfly is denoted by $BF(r)$. See Figure 1.

The butterfly network has provided new challenges and opportunities to the society. It has conveyed a unique communication to the world that it maintains data centre in the clouds and further reduces the packet loss. Network coding has been applied in wireless communication, data storage, and channel coding and computer networks to name a few. Its hypothetical consequences have been followed in mathematics, physics, and biology with ground breaking results. The butterfly network reduces the cost and network diameter. The main purpose of this technology is lightweight; it requires fewer resources and offers high security through increased impulsiveness [11].

Theorem 2.1 *Let G be a butterfly network of dimension 2, then $ipp(G, P_3) = 2$.*

Proof. Let G be the 2 dimensional butterfly network with even number of vertices. It is clear that $ipp(G) > 1$. The butterfly on 12 vertices has four 4-cycles, c_1, c_2, c_3 and c_4 and the vertex sets of these four 4-cycles effect a partition of $V(G)$. See Figure $2(a)$. The consecutive packing in these cycles are perfect. The cycles c_1, c_2, c_3 and c_4 are connected together with a single vertex common to each other. The consecutive packing in these cycles should be placed in different partitions V_1 and V_2 respectively. Thus V_1 and V_2 induce a P_3-packing partition of V(G). This implies that $ipp(BF(2)) = 2$.

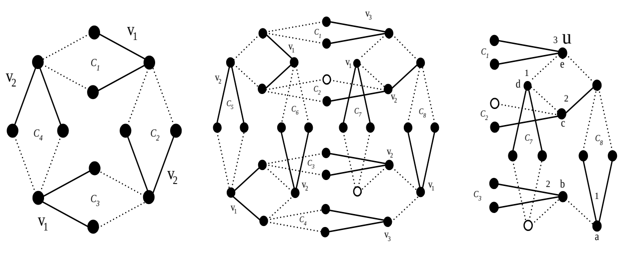


Figure 2: (a) Induced P_3-packing 2-partition of $BF(2)$ (b) Induced P_3-packing 3-partition of $BF(3)$, (c) Induced subgraph of $BF(3)$

Theorem 2.2 *Let G be the butterfly graph of dimension $n \ge 3$. Then $ipp(G) = 3$.*

Proof. We prove the result by induction on the dimension of the butterfly network. $BF(3)$ consists of two copies of butterfly network of dimension 2 say $BF(2')$ and $BF(2'')$. $BF(2')$ is partitioned as shown in Figure $2(a)$. The packing pattern of $BF(2'')$ differs from $BF(2')$, in order to cover the middle vertices that lie between $BF(2')$ and $BF(2'')$. See Figure $2(b)$. Consider the induced subgraph H', of $BF(3)$. Label the P_3-packing in H' as shown in Figure $2(c) a, b, c, d$ and e. Without loss of generality suppose $(a,d) \in V_1$ and $(b,c) \in V_2$, then $e \notin V_1$ and $e \notin V_2$. In H',vertex u cannot receive label as 1 or 2. The labeling shown in Figure $2(c)$, implies $ipp(H') = 3$. Hence $ipp(G, P_3) = 3$. Therefore $ipp(BF(3)) = 3$.

Let us assume that the result is true for butterfly network of dimension n. BF(n+1) contains two copies of $BF(n')$ and $BF(n'')$. By induction hypothesis $BF(n')$ is partitioned as $BF(n)$ and $BF(n'')$ is partitioned as $BF(n)$ with the condition V_1 being labeled as V_2 or V_3 , V_2 being labeled as V_1 or V_3 and V_3 being labeled as V_1 or V_2. Since the label of $BF(3)$ form an induced P_3-packing 3-partition , by induction hypothesis $BF(n')$ and $BF(n'')$ form an induced P_3-packing 3-partition. Therefore $ipp(BF(n)) = 3$.

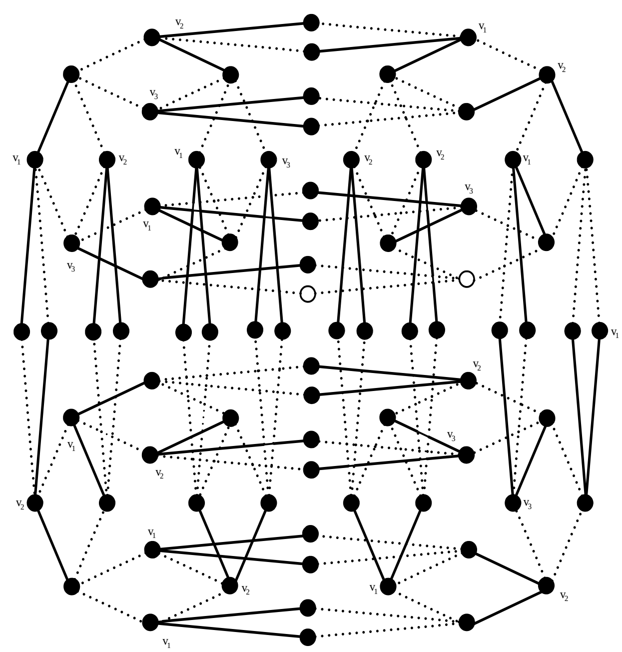


Figure 3: Induced P_3-packing 3-partition of $BF(4)$

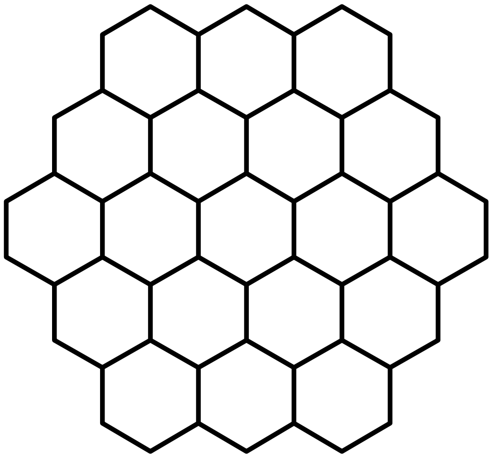
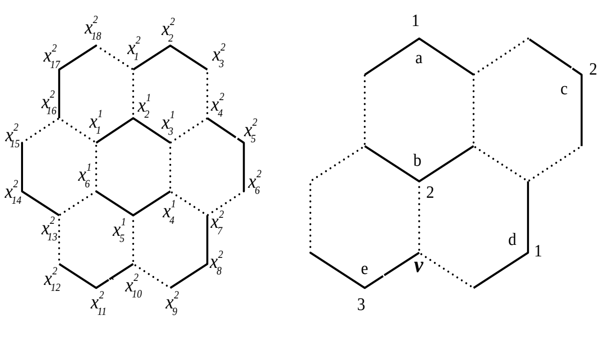


Figure 4: Structure of the Honeycomb Network

III. HONEYCOMB NETWORKS

A honeycomb network can be built in various ways. A high level honeycomb network can be constructed from a low level one. A unit honeycomb network is a hexagon, denoted by $HC(1)$. Honeycomb network of size 2 denoted $HC(2)$, can be obtained by adding six hexagons around the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ can be obtained from $HC($n - 1) by adding a layer of hexagons around the boundary edges of $HC(n-1)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2$ - $3n$ respectively. IfC_n^o denotes the outer cycle of $HC(n)$, then the number of vertices in C_n^o is $12n$ - 6 [10]. See Figure 4.



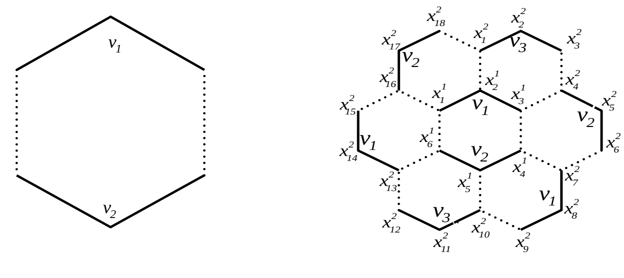
(a) (b)

Figure 5: (a) The packing of Honeycomb Network $HC(2)$, (b) The partitioning of the induced subgraph H'

Theorem 3.1 *Let G be the honeycomb network $HC(n)$. Then* $ipp(G)$ = 3

Proof. Let G be the honeycomb network. Consider the induced subgraph H' of $HC(n)$. The induced subgraph H' is packed with 5 vertex disjoint path of length 2, leaving out one vertex unsaturated. Label vertices in H' as shown in Figure $5(b)$. We claim that ipp(H') = 3. Suppose on the contrary that V_1, V_2 form an induced P_3-packing 2 - partition of H'. Label vertices in V_1 as 1 and V_2 as 2. Consider (a, b, c, d, e). Without loss of generality suppose $(a, d) \in V_1$ and $(b, c) \in V_2$, then $e \notin V_1$ and $e \notin V_2$. In H' ,vertex v cannot receive label as 1 or 2 The labeling shown in Figure $5(b)$ implies ipp(H') = 3. Hence $ipp(G, P_3)$ = 3.

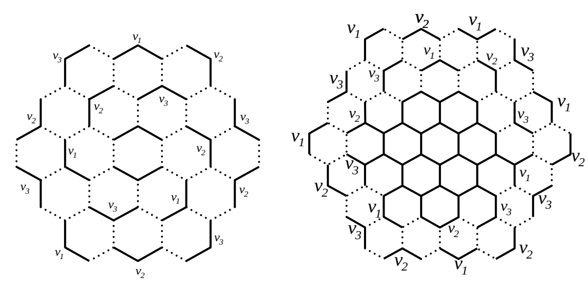
We now give a procedure and its proof of correctness to show that the proof obtained in Theorem 3.1 is sharp.



(a) *(b)*

Figure 6: (a) Induced P_3-packing 2-partition of $HC(1)$ (b) Induced P_3-packing 3-partition of $HC(2)$.

Procedure Induced Packing Partition HC(n)
Input: A honeycomb network G of dimension n and $H \simeq P_3$.
Algorithm:



(a) (b)

Figure 7: (a) Induced P_3-packing 3-partition of $HC(3)$ (b) Induced P_3-packing 3-partition of $HC(4)$.

(i) Select the hexagon $HC(1)$. In Figure $6(a)$, find P_3-packing of size 2 and denote its partition as V_1, V_2.
(ii) Having selected P_3-packing in $HC(1)$, select the P_3-packing among the six hexagons, each contains a vertex adjacent to a vertex in $HC(1)$, where $HC(1)$ have been already selected and partitioned as V_1 and V_2. In Figure $6(b)$, denote the partition as V_3, when there is no possibilities of partitioning as V_1 and V_2 in $HC(2)$.
(iii) Repeat (ii) for n dimensions of honeycomb network.See Figure 7: (a) and (b).

Output: There exists a perfect H-packing of 3-partition of honeycomb network where $H \simeq P_3$.

Proof of correctness: The selection process (iii) and the copies of P_3 selected by the procedure implies that the Honeycomp network are vertex disjoint. The algorithm covers all vertices of Honeycomb. Let V_i be the set of all vertices that receive label i, $i = 1,2,3$. For $u \in V_i$, $|N(u) \cap V_i|$ = 2, i = 1,2,3. Therefore $ipp(G, P_3)$ = 3.

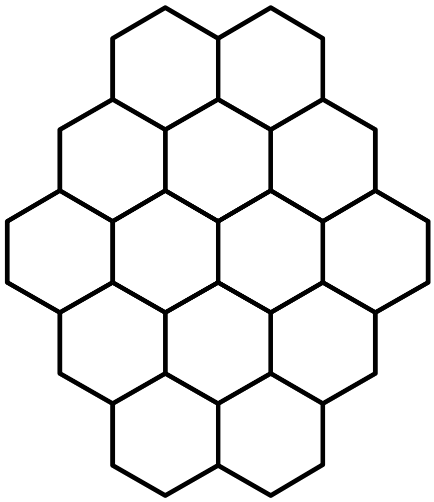


Figure 8: Circum-pyrene (1)

IV. CIRCUM PYRENE

Pyrene is an alternante polycyclic aromatic hydrocarbon (PAH) and consists of four fused benzene rings, resulting in a large aromatic system. It is a colorless or pale yellow solid which forms during incomplete combustion of organic materials and therefore can be isolated from coal tar together with a broad range of related compounds. In the last four decades, a number of research works have been reported on both the theoretical and experimental investigation of pyrene concerning its electronic structure. Like most PAHs, pyrene is used to make dyes, plastics and pesticides. Circumscribing pyrene ($C_{16}H_{10}$) gives circum-pyrene ($C_{42}H_{16}$). Inductively, circum-pyrene (n) is obtained from circum-pyrene ($n-1$) by adding a layer of hexagons around the boundary of circum-pyrene ($n-1$) [8]. See Figure 8.

Theorem 4.1 *Let H' be the induced subgraph of circum pyrene, then its P_3-packing k-partition number is* 3.

Proof. Consider the induced subgraph H' of G. The induced subgraph H' is packed with 5 vertex disjoint path of length 2, leaving out one vertex unsaturated. Label vertices in H' as shown in Figure9(a). We claim that $ipp(H')$ = 3. Suppose on the contrary that V_1, V_2 form an induced P_3-packing 2 -partition of H'. Label vertices in V_1 as 1 and V_2 as 2. Consider (x_1, x_2, x_3), (x_4, x_5, x_6), (x_7, x_8, x_9), (x_{10}, x_{11}, x_{12}), (x_{13}, x_{14}, x_{15}) and (x_{16}). Without loss of generality suppose (x_1, x_2, x_3), (x_7, x_8, x_9) $\in V_1$. (x_4, x_5, x_6), (x_{10}, x_{11}, x_{12}) $\in V_2$. Then (x_{13}, x_{14}, x_{15}) $\notin V_1$ or V_2, where the vertex x_{16} is left uncovered. The labeling shown in figure 9(a) implies ipp(H') = 3. In G, vertex u cannot receive label as 1 or 2. Hence $ipp(G, P_3)$ = 3.

Procedure Induced Packing Partition Circum Pyrene
Input: A Circum Pyrene G of dimension n and $H \simeq P_3$.
Algorithm:
(i) Label the three vertices x_4, x_5 and x_6 in Figure 9(a) as 2 if the vertex x_3 and x_7 is labeled as 1.
(ii) Label the three vertices x_{10}, x_{11} and x_{12} in Figure 9(a) as 2 if the vertex x_9 and x_{13} is labeled as 1 or 3 and 3 or 1 respectively.
(iii) Label the packing from vertices x_1, x_2, x_3 ... x_n in Figure 9(b) with labels 1, 2 and 3, such that no two adjacent packing receives the same label.
Output: induced P_3-packing partition number for Circum pyrene is 3.
Proof of correctness: The vertices that receive label 1, 2 or 3 are in V_1, V_2 and V_3 respectively. For any $x \varepsilon V_i$, i= 1, 2, 3 exactly one vertex in $N(x) \in V_i$. Therefore $ipp(G)$ = 3.

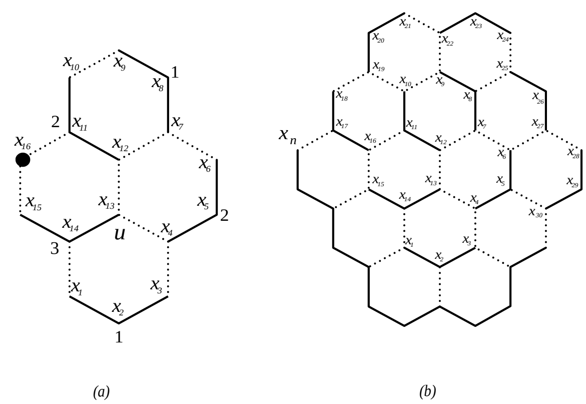


Figure 9: (a) Induced Subgraph H' (b) Induced P_3-packing 3-partition of Circum Pyrene

V. CONCLUSION

In this paper we investigate the induced H-packing k-partition number is 3 for Butterfly network, Honeycomb Networks and Circum Pyrene. It would be an interesting line of research to determine the induced H-packing k-partition number for other chemical graphs.

REFERENCES

[1] A. Al Mutairi, Bader Ali and D. Paul Manuel, "Packing in Carbon Nanotubes", Journal of Combinatorial Mathematics and Combinatorial Computing , 92, 2015, 195 - 206.

[2] R. Bar-Yehuda, M. Halldorsson, J. Naor, H. Shachnai and I. Shapira, "Scheduling split intervals", in: Proc. Thirteenth Annu. ACM - SIAM Symp, 2002,732 - 741.

[3] R. Bejar, B. Krishnamachari, C. Gomes and B. Selman, "Distributed constraint satisfaction in a wireless sensor tracking system", Workshop on Distributed Constraint Reasoning, Internat. Joint Conf. on Artificial Intelligence, 2001.

[4]A. Felzenbaum, "Packing lines in a hypercube", Discrete Mathematics 117, 1993, 107 - 112.

[5] P. Hell and D. Kirkpatrick, "On the complexity of a generalized matching problem", in: Proc. Tenth ACM Symp, 1978, 309 - 318.

[6] A. Muthumalai, I. Rajasingh and A. S. Shanthi, "Packing of Hexagonal Networks", Journal of Combinatorial Mathematics and Combinatorial Computation, 79, 2011, 121 - 127.
[7] D. Paul Manuel, I. Mostafa Abd-El-Barr, I. Rajasingh, Bharati Rajan, "An efficient representation of Benes networks and its applications", Journal of Discrete Algorithms, 6, 2008, 11 - 19.
[8] J. Quadras, K. Balasubramanian, K.A. Christy, "Analytical expressions for Wiener indices of n-circumscribed peri-condensed benzenoid graphs", Journal of Mathematical Chemistry, 54, 2016, 823 - 843.
[9] S.M.J. Raja, A. Xavier, I. Rajasingh, "Induced H-packing k-partition problem in interconnection networks", International Journal of Computer Mathematics: Computer Systems Theory. 2, 2017 136 - 146.
[10] I. Rajasingh, A. Muthumalai, R. Bharati and A. S. Shanthi, "Packing in honeycomb networks", Journal of Mathematical Chemistry, 50, 5, 2012, 1200 - 1209.
[11] H. Liu, L. Xie, J. Liu, L. Ding, "Application of Butterfly Clos-Network in Network-on-Chip", The Scientific World Journal, 2014, 1–11.