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Kernels in Mycielskian of a Digraph

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Abstract — A kernel J of a digraph D is an independent set of vertices of D such that for every vertex $w \in V(D) \setminus J$ there exists an arc from w to a vertex in J. The *Mycielskian* $\mu(D)$ of a digraph $D = (V, A)$ is the digraph with vertex set

$$V \cup V' \cup \{u\},$$

where $V' = \{v' : v \in V\}$, and the arc set

$$A \cup \{(x, y') : (x, y) \in A\} \cup \{(x', y) : (x, y) \in A\} \cup \{(x', u) : x' \in V'\} \cup \{(u, x') : x' \in V'\}.$$

In this paper, we have proved that, for any digraph D, the Mycielskian of $\boldsymbol{D}$, $\boldsymbol{\mu}(\boldsymbol{D})$, contains a kernel.

***Keywords* —** Kernel, Mycielskian of a digraph

I. INTRODUCTION

For notation and terminology, in general, we follow [1].

Let $D = (V, A)$ denote a digraph, where V and A are the vertex set and arc set of D, respectively. All the digraphs considered here are finite and has no loops and no parallel arcs. A loop is an arc of the form (v, v), where v is a vertex of D. Arcs of D of the form $e_1 = (u, v)$ and $e_2 = (u, v)$ are called parallel arcs of D.

For an arc (u, v), the first vertex u is its tail and the second vertex v is its head. For $(u, v) \in A$, we also use either $u \to v$ or $v \leftarrow u$. If there is a directed path from u to v in D, then the distance, $d_D(u, v)$, from u to v is the length of any shortest directed (u, v)- path in D. If S is a non-empty subset of V, then the subdigraph, $D[S]$, induced by S is the digraph with vertex set S and those arcs of D which join vertices of S.

A set $X \subseteq V$ is said to be *independent* if the subdigraph induced by X has no arcs.

A set $Y \subseteq V$ is said to be *absorbent* if for each $z \in V \setminus Y$, there exists $x \in Y$ such that $d_D(z, x) = 1$.

A set J of vertices in a digraph D is said to be a *kernel* if J is both independent and absorbent. A digraph D is called *kernel-less* if it has no kernel.

Let D be a digraph and let $O = \{v \in V(D) : d_D^+(v) = 0\}$. Then, observe that, any kernel of D, if it exists, contains O.

Let X and Y be non-empty subsets of $V(D)$. If for every $x \in X$ and every $y \in Y$ we have that $(x, y) \in A(D)$, we will write $X \to Y$. When $X = \{u\}$ for some $u \in V(D)$, we will simply write $u \to Y$ and analogously we write $X \to v$ if $Y = \{v\}$.

For a vertex $x \in V(D)$, define the *out-neighborhood* of x in D as the set $N_D^+(x) = \{y \in V(D) : (x, y) \in A(D)\}$. The elements of $N_D^+(x)$ are called the *out-neighbors* of x, and the *out-degree* of x, $d_D^+(x)$, is the number of out-neighbors of x. Similarly, for a vertex $x \in V(D)$, define the *in-neighborhood* of x in D as the set $N_D^-(x) = \{y \in V(D) : (y, x) \in A(D)\}$. The elements of $N_D^-(x)$ are called the *in-neighbors* of x, and the *in-degree* of x, $d_D^-(x)$, is the number of in-neighbors of x.

By convention, $\delta^-(D), \delta^+(D), \Delta^-(D)$ and $\Delta^+(D)$ denote, respectively, the minimum in-degree, the minimum out-degree, the maximum in-degree and the maximum out-degree of the digraph D.

In [2], Von Neumann and Morgenstern introduced the concept of kernel of a digraph in the context of Game Theory. Kernels have found many applications, for instance in game theory, mathematical logic, computational complexity, coding theory and in list edge-colouring of graphs, see p. 119 of [3].

The concept of kernel seems to have a lot of potential modeling real life situations, for example, if we have the map of a region modeled by a digraph (where vertices are locations and arcs are streets or highways between locations), and we want to find an optimal set of locations to build a service center (e.g. hospitals or schools) easily accessible to the whole population, a possible solution to this problem is to find a kernel in the digraph. The absorbence of the kernel will guarantee that from every location not in our kernel, one of the selected locations in the kernel can be easily reached.

Finding kernels in special classes of digraphs seems to be difficult. Chvátal (see [3], p. 204) proved that the problem of deciding whether a given digraph has a kernel is NP-complete. A nice survey on kernel is available (see [4]).

The problem of the existence of a kernel in a given digraph has been studied by several authors in particular by Richardson [5,6], Duchet and Meyniel [7], Duchet [8,9], Galeana-Sánchez and Neumann-Lara [10]

The special classes of digraphs that we concentrate on here is Mycielskian of a digraph. In Section 2, we have proved that the Mycielskian of a digraph has a kernel irrespective of the choice of the digraph.

II. A RESULT ON MYCIELSKIAN OF A DIGRAPH

Let $D = (V, A)$ be a digraph. The *Mycielskian* $\mu(D)$ of D is defined, in [12], as follows: $\mu(D)$ is the digraph with vertex set

$$V \cup V' \cup \{u\},$$

where $V' = \{v' : v \in V\}$, and the arc set

$$A \cup \{(x, y') : (x, y) \in A\} \cup \{(x', y) : (x, y) \in A\} \cup \{(x', u) : x' \in V'\} \cup \{(u, x') : x' \in V'\}.$$

The vertex x' is called the *twin* of the vertex x and the vertex u is called the *root* of $\mu(D)$. For example, see Figure 2.

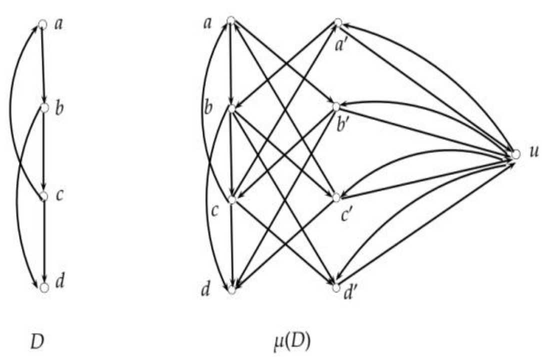

Figure 2

Theorem 2. *For any digraph D, the Mycielskian of D, $\mu(D)$, contains a kernel.*

Proof. For $X \subseteq V(D)$, set $X' = \{x' : x \in X\}$. If $\delta^+(D) \geq 1$, then $V(D)'$ is a kernel of $\mu(D)$. (By the definition of $\mu(D)$, $V(D)'$ is independent in $\mu(D)$. For every $v \in V(D)$, $u \to v'$, where v', the twin of v, is in $V(D)'$. If $x \in V(D)$, then, as $\delta^+(D) \geq 1$, there exists $y \in V(D)$ such that $x \to y$ in D, and hence $x \to y'$ in $\mu(D)$, where y', the twin of y, is in $V(D)'$.)

If D contains a kernel, then, let J be a kernel of D. Now $J \cup \{u\}$ is a kernel of $\mu(D)$. (As J is independent in D, $J \cup \{u\}$ is independent in $\mu(D)$. For every v', the twin of v, belonging to $V(D)'$, $v' \to u$. If $x \in V(D) \setminus J$, then, as J is a kernel of D, there exists $y \in J$ such that $x \to y$ in D, and hence $x \to y$ in $\mu(D)$.)

Hence, assume that D contains no kernel and $\delta^+(D) = 0$.

We successively construct subsets $A_1, B_1, A_2, B_2, A_3, B_3, \ldots$ of $V(D)$ as follows: Finally, using A_i's and B_i's, we construct J_0, a kernel of $\mu(D)$. Denote by D_i, the subdigraph of D, obtained from D by deleting the set of vertices in $(A_1 \cup \ldots \cup A_{i-1}) \cup (B_1 \cup \ldots \cup B_{i-1})$. Set $A_1 = \{a_1 \in V(D) : d_D^+(a_1) = 0\}$ and $B_1 = N_D^-(A_1) = \{b_1 \in V(D) | \, b_1 \to a_1$ for some $a_1 \in A_1\}$. For $i \geq 2$, set $A_i = \{a_i \in V(D_i) : d_{D_i}^+(a_i) = 0\}$ and $B_i = N_{D_i}^-(A_i) = \{b_i \in V(D_i) | \, b_i \to a_i$ for some $a_i \in A_i\}$. As $\delta^+(D) = 0$, $A_1 \neq \emptyset$. Also, for each i, A_i is independent in D_i and hence A_i is independent in D. As D is finite, there exists j such that $A_{j-1} \neq \emptyset$ and $A_j = \emptyset$.

Set $J_0 = (A_1 \cup \ldots \cup A_{j-1}) \cup (A_1 \cup \ldots \cup A_{j-1})' \cup K'$, where $K = V(D) \setminus [(A_1 \cup \ldots \cup A_{j-1}) \cup (B_1 \cup \ldots \cup B_{j-1})]$.

Claim 1. $A_1 \cup \ldots \cup A_{j-1}$ is independent in D.

Otherwise, $A_1 \cup \ldots \cup A_{j-1}$ is not independent in D. Then there exist r and s with $1 \leq r < s \leq j-1$ and either $a_r \to a_s$ or $a_s \to a_r$, where $a_r \in A_r$ and $a_s \in A_s$. As $r < s$, $a_s \in V(D_r)$. As $d_{D_r}^+(a_r) = 0$, $a_r \to a_s$ is impossible. If $a_s \to a_r$, then $a_s \in B_r$, a contradiction.

Claim 2. $K \neq \emptyset$.

Otherwise, $K = \emptyset$. Then $A_1 \cup \ldots \cup A_{j-1}$ is a kernel of D, a contradiction. (If $v \in V(D) \setminus (A_1 \cup \ldots \cup A_{j-1}) = B_1 \cup \ldots \cup B_{j-1}$, then $v \in B_i$, $i \in \{1, \ldots, j-1\}$. Hence, there exists $a_i \in A_i$ such that $v \to a_i$. This shows that $A_1 \cup \ldots \cup A_{j-1}$ is absorbent in D.)

Claim 3. J_0 is independent in $\mu(D)$.

Both the sets $A_1 \cup \ldots \cup A_{j-1}$ and $(A_1 \cup \ldots \cup A_{j-1})' \cup K'$ are independent sets in $\mu(D)$. If the claim is not true, then there exist $x \in A_1 \cup \ldots \cup A_{j-1}$ and $y' \in (A_1 \cup \ldots \cup A_{j-1})' \cup K'$ such that, in $\mu(D)$, either $x \to y'$ or $y' \to x$. Thus, there exist $x \in A_r$, $1 \leq r \leq j-1$, and $y \in (A_1 \cup \ldots \cup A_{j-1}) \cup K$ such that, in D, either $x \to y$ or $y \to x$. Since $A_1 \cup \ldots \cup A_{j-1}$ is independent in D, $y \notin A_1 \cup \ldots \cup A_{j-1}$. Hence, $y \in K$, and therefore $y \in V(D_r)$. As $d_{D_r}^+(x) = 0$, $x \to y$ is impossible. If $y \to x$, then $y \in B_r$, a contradiction.

Claim 4. J_0 is absorbent in $\mu(D)$.

Choose $w \notin J_0$.

If $w = u$, then $u \to K'$. If $w = b_r \in B_r$, $1 \leq r \leq j-1$, then there exists $a_r \in A_r$ such that $b_r \to a_r$. If $w = b_r' \in B_r'$, $1 \leq r \leq j-1$, then $b_r \in B_r$ and so there exists $a_r \in A_r$ such that $b_r \to a_r$, and hence $b_r' \to a_r$.

As $A_j = \emptyset$, $\delta^+(D[K]) \geq 1$.

If $w \in K$, then, as $\delta^+(D[K]) \geq 1$, there exists $k \in K$ such that $w \to k$, and therefore $w \to k'$, where k', the twin of k, is in K'.

This completes the proof.

III. Conclusion and Future Scope

This paper extends the known classes of digraphs containing kernel. One can investigate some other unexplored classes of digraphs in the aspect of presence or absence of kernel.

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