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Algorithms for Variational Inequalities

Poonam Mishra [1*], Shailesh Dhar Diwan [2]

[1]Department of Applied Mathematics, Amity University, Raipur, Chhattisgarh, India
[2]Dept. of Applied Mathematics, Government Engineering College, Raipur, Chhattisgarh, India

Corresponding Author: ppoonam22@gmail.com

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***Abstract*—** Variational inequalities are studied in various models for a large number of mathematical, physical, economics, finance, optimization, game theory, engineering and other problems(see[1],[2],[12], [14],[15], [21]). The fixed point formulation of any variational inequality problem is not only useful for existence of solution of the variational inequality problem, but it also provides the facility to develop algorithms for approximation of solution of variational inequality problem. A lot of research has been carried out to develop various iterative algorithms to find solution of a variational inequality problem. In this paper, we have studied various algorithms or methods used for solving Variational inequality problems and studied the developments of such methods and compared their convergence rate . Our result helps in understanding the development in iterative algorithms for VI.

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Keywords: Proximal Point Algorithm, KKT based method ,Linear Approximation method, strong convergence, variational inequality, Projection based methods.

I. INTRODUCTION

The variational inequality problem has emerged as an important tool to study a variety of problems in economics, optimization, operations research, structural analysis and many engineering sciences. The variational inequality problem was first introduced and studied by Stampacchia (see [5]), since then it has been studied by various authors and efforts have been made to find suitable methods for solving variational inequality problem

The algorithms for solving VI(K, F) can be classified into several categories depending upon which formulation a method exploits. There are methods based on KKT conditions , gap/merit functions, interior and smoothing methods, and projection based methods.

Rest of the paper is organized as follows, Section I contains the introduction of Variational Inequality and the various algorithms used for solving VI , Section II contains the related developments in algorithms, Section II,III IV ,V and VI contains the major developments in different algorithm for VI viz. Linear Approximation, KKT based, Proximal Point and Projection based algorithms and compared their convergence rate and conditions for convergence. Section VI describes results and discussions of our study. Section VII contain concludes research work with future directions.

II. ALGORITHMS FOR VI

The algorithms for solving variational inequalities can also be categorized based on the subproblems that are solved in each iteration. A general approach to solving VI(K, F) consists of
creating a sequence $\{x_k\} \subset K$ such that each x^{k+1} solves VI(K, F^k),

$$\left\langle F^k(x^{k+1}), y - x^{k+1} \right\rangle \geq 0 \text{ for all } y \in K, \quad (2.1)$$

where $F^k(.)$ is some approximation to F(x). F^k can be linear or nonlinear.

III. LINEAR APPROXIMATION BASED METHODS

A linear F^k is of the form

$$F^k(x) = F(x^k) + \left\langle A(x^k), x - x^k \right\rangle. \quad (3.1)$$

As described by Harker and Pang[3], different choices of $A(x^k)$ lead to different methods.
1. Newton's method: $A(x^k) = \nabla F(x^k)$.
2. Quasi Newton method: $A(x^k) \approx \nabla F(x^k)$.

3. Linearized Jacobi method: $A(x^k) = D(x^k)$, where $D(x^k)$ is the diagonal part of $\nabla F(x^k)$.
4. Successive over relaxation: $A(x^k) = T(x^k) + D(x^k) / \eta^*$, where $T(x^k)$ is the upper or lower triangular part of $\nabla F(x^k)$ and η^* is a parameter in (0, 2).
5. Symmetrized Newton: $A(x^k) = ½\{\nabla F(x^k) + \nabla F(x^k)^T\}$.
6. Projection method: $A(x^k) = G$, a symmetric positive definite matrix.

The convergence of these methods depends on x^* being a regular solution to VI(K, F).

Definition 3.1. (Robinson[4)] Let x^* be a solution to VI(K, F), and F be differentiable at x^*.
Then x^* is called a regular solution if there exists a neighborhood N of x^* and a scalar $\alpha > 0$ such that for every y with $||y||_2 < \alpha$, there is a unique vector $x(y) \in N$, Lispchitz continuous with respect to y, that solves the perturbed linearized VI(K, F^y) with $F^y : R^n \to R^n$ defined as

$$F^y(x) = F(x^*) + y + \langle \nabla F(x^*), x - x^* \rangle.$$

Let the set K be defined as in (2.1) with g_i, h_j being twice continuously differentaible for each i and j, and F being once continuously differentiable. Let $x^* \in SOL(K, F)$. Suppose that the following conditions hold.
1. There exist vectors $\mu^* \in R^l$, and $\lambda^* \in R^m$, such that (x^*, μ^*, λ^*) satisfy the KKT conditions for VI(K, F).
2. Linear independence constraint qualification (LICQ) holds at x^*. That is, the vectors $\{\nabla g_i(x^*)$:
$i \in I_+ \cup I_0$; $\nabla hj(x^*)\}$ are linearly independent, where $I_+ = \{i : \lambda^*_i > 0\}$ and $I_0 = \{i : g_i(x^*) = 0, \lambda^*_{i=} 0\}$
3. The second order condition

$$\left\langle z, [\nabla F(x^*) + \sum_{i=1}^{l} \mu_i^* \nabla h_i^2(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i^2(x^*)] z \right\rangle > 0 \quad (2.4)$$

holds for all $z \neq 0$ such that,

$$\langle z, \nabla g_i(x^*) \rangle = 0 \quad \forall i \in I_+,$$
$$\langle z, \nabla h_i(x^*) \rangle = 0 \quad \forall j = 1, 2, ..., l.$$

Then x^* is a regular solution to VI(K, F) (Robinson[4]). If x^* is a regular solution to VI(K,F), then there exists a neighborhood N of x^* such that Newton's method converges to x^* as long as it starts from an initial point $x^0 \in N$ (Josephy[7]). Furthermore, if $\nabla F(x^*)$ is Lipschitz continuous around x^*, then the convergence rate is quadratic. But Newton's method for solving VI(K, F) suffers from the following drawbacks.
1. $\nabla F(x^*)$ needs to be evaluated at every step.
2.Each iteration requires solving a variational inequality subproblem.
3. The method converges only if the initial iterate is close enough to a solution.

Quasi Newton methods overcome the first drawback of Newton's method. For instance, secant methods (Josephy[6]) update the matrix $A(x^k)$ in each iteration by a simple small rank matrix. Although this reduces the work of finding $\nabla F(x^k)$ at each iteration, it does not make solving the subproblems any easier. Those methods can achieve a superlinear convergence rate at best.

Other linear approximation methods, including the linearized Jacobi method, symmetrized Newton method and projection algorithms, use a symmetric matrix $A(x^k)$ at each step. In these methods,the subproblem can be formulated as an optimization problem, thereby making it amenable to various optimization algorithms. On the downside, those methods require stronger restrictions on the problem, and do not have quadratic rate of convergence. The linearized successive over relaxation method solves an LCP with a triangular matrix at each step. The following theorem summarizes the performance of the linearized Jacobi method and the symmetrized Newton's method.

Theorem 3.1. (Chan and Pang[8]) Let K be a nonempty, closed and convex subset of R^n and let F be a function from R^n to R^n.
1. Suppose that F is once continuously differentiable, x^* solves VI(K, F) and $\nabla F(x^*)$ has positive diagonal elements. Let $D(x^*)$ and $B(x^*)$ be the diagonal and off diagonal parts of $\nabla F(x^*)$ respectively. If

$$\left\| D(x^*)^{-1/2} \right\|_2 < 1; \quad (3.5)$$

then there exists a neighborhood of x^* such that the sequence generated by linearized Jacobi method is well defined and converges to x^* if it starts with an initial point within that neighborhood.
2. Suppose that F is once continuously differentiable, x^*solves VI(K, F), and that $\nabla F(x^*)$ is positive definite. Let $A(x^*)$ and $C(x^*)$ be the symmetric and skew symmetric parts of $\nabla F(x^*)$ respectively. If

$$\left\| C(x^*) \right\|_2 < \lambda_{\min}(A(x^*)), \quad (3.6)$$

where $\lambda_{\min(A(x^*))}$ denotes the least eigenvalue of $A(x^*)$, then there exists a neighbourhood of x^* such that the sequence generated by the symmetrized Newton method is well defined and converges to x^* if it starts with an initial point within that neighborhood.

Moreover, the convergence rate of each of these methods is geometric, that is, there exists a constant $r \in (0; 1)$ such that for a certain vector norm and for all k,

$\|x^{k+1} - x^*\| \le r\|x^k - x^*\|$ holds.

A class of methods applicable to VI(K, F) when K is a compact polyhedral set are the simplicial decomposition methods. Since K is a compact polyhedron, it can be expressed as the convex hull of its extreme points. At iteration k, a VI(K^k, F^k) is solved, where K^k denotes the convex hull of a subset of extreme points of K. A merit function is used to decide upon the addition or deletion of extreme points from K^k to obtain K^{k+1}. The effectiveness of the method depends on how many extreme points K has, and on the merit function used to guide the choice of extreme points at each iteration (Lawphongpanich and Hearn[9]) shows that if the gap function $\min_{y\in K}$ <F(x), y – x> is used to control K^k, and F^k = F for all iterations, then the method terminates in a finite number of major iterations if F is strongly monotone.

In what follows, we briefly describe methods based on the KKT formulations of variational inequalities, and proximal point methods.

IV. KKT BASED METHODS

Methods based on the KKT formulation of VI(K, F) try to solve systems of nonsmooth constrained equations or minimize a merit function derived from (2.1). Using the min(.,.) and the Fischer-Burmeister (FB) C-function,

$$\psi_{FB}(a,b) = \sqrt{a^2+b^2} - (a+b), \forall (a,b) \in R^2,$$

we can obtain two different equation reformulations for VI(K, F). Let

$$\phi(x,\mu,\lambda) = \begin{pmatrix} L(x,\mu,\lambda) \\ h(x) \\ C_{FB}(-g_1(x),\lambda_1) \\ \vdots \\ C_{FB}(-g_m(x),\lambda_m) \end{pmatrix},$$

$$\phi_{\min}(x,\mu,\lambda) = \begin{pmatrix} L(x,\mu,\lambda) \\ h(x) \\ \min(-g_1(x),\lambda_1) \\ \vdots \\ \min(-g_m(x),\lambda_m) \end{pmatrix}$$

A natural merit function for the KKT formulation is

$$\theta(x,\mu,\lambda) = \frac{1}{2}\langle \phi(x,\mu,\lambda), \phi(x,\mu,\lambda)\rangle,$$

where $\phi(x,\mu,\lambda)$ can be either $\phi_{FB}(x,\mu,\lambda)$ or $\phi_{\min}(x,\mu,\lambda)$. One can then try to solve the equation $\phi(x,\mu,\lambda)= 0$, or to minimize $\theta(x,\mu,\lambda)$. Algorithms based on the merit function $\theta(x,\mu,\lambda)$ can be regarded as special cases of interior point methods which use a more generic potential function $p(\phi(x,\mu,\lambda))$ to measure the improvement in each iteration. Methods based on the natural gap function for the VI are a special case of Zhu and Marcotte's general framework for solving VIs. The methods mentioned here find a descent direction for the merit or potential function at each iteration, and perform a line search routine in that direction to find the next iterate. One can refer to the book (Facchinei and Pang[10]) for details on these algorithms.

V. PROXIMAL POINT METHOD

The proximal point method is another class of solution methods for VIs. This method solves VI(K, F + ε_kI) at iteration k. Here $\{\varepsilon_k\}$ is a sequence of positive scalars going to zero, and I is the identity map. If F is monotone, then F + εI is strongly monotone. Thus each subproblem has a unique solution. (Rockafellar [11]) showed that if ε_k are chosen according to an appropriate inexact rule, then the sequence $\{x^k\}$ is bounded if and only if SOL(K, F) $\neq \phi$. Moreover, if the sequence $\{x^k\}$ is bounded, then it converges to a solution of VI(K, F).

In the following generic proximal point scheme, VI(K, $F_{c,x}$), where $F_{c,x}(y)$ = y - x + cF (y) is solved inexactly at each iteration. The algorithm uses the fact that if F is monotone then the set valued map F + N(K,.) is maximal monotone. A set valued map ϕ: $R^n \to R^n$ is (strongly) monotone if there exists a constant c (>) ≥ 0 such that

$$\langle x-y, u-v\rangle \ge c\|x-y\|^2 \ \forall x,y \in dom(\phi)$$
$$\text{and } u \in \phi(x),\ v \in \phi(y). \tag{2.7}$$

A monotone map ϕ is maximal monotone if no monotone map ψ exists such that graph$\phi \subset$ graphϕ. The properties of maximal monotone maps ((Facchinei and Pang[10]) play an important role in the development of the algorithm being described here.

Algorithm 5.1 : Proximal point method for VIs

Initialization: Choose $x^0 \in R^n$, $c_0 > 0$, sequences $\{\varepsilon_k\}$, $\{\alpha_k\}$, and $\{c_k\}$ as required by the previous theorem. Set k = 1, loop = 0.
while loop = 0 **do**
if $x_k \in$ SOL(K, F) **then**
Set loop = 1.
else
Find w_k such that $\|w^k - Jc_{kT}(x^k)\| \le \varepsilon_k$.
Set $x^{k+1} = x^k + \alpha_k(w^k - x^k)$,

Set k = k + 1.
Select c_k,ε_k, and α_k.
end if
end while

If the VI has a solution, the algorithm converges to it. Otherwise the sequence generated by thelgorithm is unbounded.

VI. PROJECTION BASED METHODS

We observed that x solves VI(K, F) if and only if

$$x = \prod_{K,D}(x - D^{-1}F(x)) \qquad (6.1)$$

where $\prod_{K,D}$ is the skewed projector onto K defined by a n ×n positive definite matrix D.

If the projection map defined in (6.1) is a contraction[5], the sequence $\{x^k\}$,k=0 to ∞ defined as $x = \prod_{K,D}(x_k - D^{-1}F(x_k))$ converges to its fixed point irrespective of the choice of x_0.

Theorem 6.1. ((Facchinei and Pang, [10]) Let K be a closed and convex subset of Rn and F :K → Rn be μ monotone and Lipschitz continuous with constant L. If

$$L^2\lambda_{max}(D) < 2\,\mu\lambda^2{}_{min}(D); \qquad (6.2)$$

then the mapping $\prod_{K,D}(x - D^{-1}F(x))$ is a contraction from K to K with respect to the norm||.||. Moreover, the sequence $\{x^k\}$ generated by the iterations

$$x^{k+1} = \prod_{K,D}(x^k - D^{-1}F(x^k)) \qquad (6.3)$$

starting from any $x^0 \in K$, converges to the solution of the VI(K, F) with a linear rate of convergence.

One issue with the standard projection method is that it requires knowledge of constants characterizing Lipschitz continuity and strong monotonicity. The extragradient method requires a slightly weaker assumption on F, that is, F needs to be pseudomonotone. It requires two pojection calculations in each iteration:

$$x^{k+1/2} = \prod_K(x^k - \tau F(x^k)),$$

$$x^{k+1} = \prod_K(x^k - \tau F(x^{k+1/2}))$$

The extragradient method still requires knowledge of the Lipschitz constant, L, of F, since it converges only if τ < L.

Many recent developments have taken place in projection based methods[1],[14]

VII. RESULTS AND DISCUSSIONS

In this pap er, we have studied various algorithms for solving Variational Inequality Problem and analysed various constraints for convergence of the solution. The convergence of various models based on Linear Approximation, KKT, Proximal Point and Projection are foused and analysed. Recent results and advancements in this regard is important and thus stimulates the strong convergence of VI problems.

VIII.CONCLUSION

Through the study on various methods or algorithms developed so far for solving variational inequality problem we establish that various constraints exists for finding the solution of VI and still the developments in this direction are going on ([15], [16], [17], [18], [19],[20]) as variational inequality is emerging as one of the most important field of study due to its applications in all emerging fields.

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Authors Profile

Ms.Poonam Mishra pursed Master of Science from Pt.Ravishankar Shukla University, Raipur,India.She is M.Phil in Mathematics and completed her M.B.A in Operations Management. She is currently pursuing Ph.D. and working as Assistant Professor in Department of Applied Mathematics,Amity School of Engineering & Technology,Amity University Chattisgarh,India.She has published more than 15 research papers in reputed international journals including Thomson Reuters and international conferences .Her main research work focuses on Variational Inequalities,Fixed Point Theory, Optimization and computational Mathematics.She has more than 12 years of teaching experience .

Dr. Shailesh Dhar diwan is currently working as Associate Professor in Government Engineering College,Raipur Chhattisgarh ,India.He has more than 20 years of experience and has guided more than 5 research scholars.His main research fosuses on Fixed Point theory,Approximation theory and Fuzzy Logic.

