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Further Results on Sum *Number and Mod Sum* Number of Graphs

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Abstract**—** In this paper we establish that the graphs $K_n - E(K_r)$, $K_{n,n}$ for $n \geq 2$, $K_{n,n} - E(nK_2)$ for $n \geq 2$, $P_n \odot K_1$ for $n \geq 2$ and $C_n \odot K_1$ for $n \geq 4$ possesses sum* and modsum* labelings and find their sum * and mod sum* numbers.

Keywords: Sum* graphs, Sum* number, Mod sum* graphs and Mod sum* number.

I. INTRODUCTION

The graphs considered here are finite, connected, undirected and simple. The notations and terminologies involving graph theory may be found in [2] .The study undertaken in this paper involves sum* and mod sum* labeling of graphs. The objective of this work is to explore and identify some new classes of graphs that exhibit sum* and mod sum* labeling. In this paper we use a methodology which fundamentally involves formulation and subsequent mathematical validation. Sum* and mod sum* labeling concepts have been used in the problems involving relational database management. We recapitulate some important definitions useful for the present investigation. The concept of a sum graph was introduced by Harary in 1990 [3]. Let N be the set of positive integers. The sum graph *G*(S)* of a finite subset $S \subset N$ is the graph (S, E) with $uv \in E$ if and only if $u + v \in S$. A graph G is said to be a sum graph if it is isomorphic to the sum graph *G(S)* for some $S \subset N$.The *sum number* $\sigma(G)$ of a graph G is the least number r of isolated vertices rK_1 such that $G \cup rK_1$ is a sum graph. The concept of mod sum graph was introduced by Boland et al. [4] in 1990. A mod sum graph is a sum graph with $S \subset Z_m \backslash \{0\}$ and all arithmetic performed modulo m where $m \geq |S| + 1$. The *mod sum number* $\rho(G)$ of graph G is the least number r of isolated vertices rK_1 such that $G \cup rK_1$ is a mod sum graph.

The notion of sum* graphs and mod sum* graphs were introduced by Sutton in 2001 [1]. A graph $G = (V_p \cup V_i, E)$ a sum* graph of $G_p = (V_p, E_p)$ if there is an injecting labeling λ of the vertices of G with distinct nonnegative integers with the property that $uv \in E_p$ if and only if $\lambda(u) + \lambda(v) = \lambda(z)$ for some vertex $z \in G$. The sum* number $\sigma^*(G_p)$ of G_p is the minimum cardinality of a set of new vertices such that there exists a sum* graph of G_p on the set of vertices $V_p \cup V_i$. Sum* graphs are generalization of sum graphs. Sutton shows that every graph is an induced sub graph of a connected sum* graph. A graph $G = (V_p \cup V_i, E)$ is a mod sum* graph of $G_p = (V_p, E_p)$ if there exists a positive integer $\boldsymbol{z}$ and a labeling λ of the vertices of G with distinct elements from $\{0, 1, 2, \ldots, z-1\}$ so that $uv \in E_p$ if and only if $(\lambda(u) + \lambda(v))(mod\ z)$ is the label of vertex of G, where V_i is an incidental vertex set of a summable graph that is not vertex of the primary graph G_p.The mod sum* number $\rho * (G_p)$ of G_p is the cardinality of the smallest set of incidentals V_i such that there exists a mod sum* graph of G_p on $V_p \cup V_i$ vertices. Mod sum* graphs are a generalization of mod sum graphs, so that all mod sum graph labeling are also mod sum* graph labeling.
Sutton in his PhD thesis [1] has obtained established that $\sigma * (K_2) = \sigma * (S_1) = 0$, $\sigma * (S_n) = 0$ for $n \geq 2$, $\sigma * (T_n) = 1$ for $n \geq 3$ and $T_n \neq S_n$, $\sigma * (C_3) = 1$, $\sigma * (C_n) = 2$ for $n \geq 4$, $\sigma * (W_n) = 2$ for $n \geq 4$, $\sigma * (F_n) = 2$ for $n \geq 3$, $\sigma * (K_n) = n - 2$ for $n \geq 3$, $\rho * (K_2) = \rho * (S_1) = 0$, $\rho * (S_n) = 0$ for $n \geq 2$, $\rho * (T_n) = 0$ for $n \geq 3$ and $T_n \neq S_n$, $\rho * (C_3) = 0$, $\rho * (C_n) = 0$ for $n \geq 4$, $\rho * (W_n) = 0$ for $n \geq 4$, $\rho * (F_n) = 0$ for $n \geq 3$, $\rho * (K_n) = 0$ for $n \geq 3$.

Here our objective and purpose is to explore more on sum* and mod sum* labelling of graphs by extending the findings of Sutton [1] and find some new classes of graphs exhibiting sum* and mod sum* labelling and find their sum* and mod sum* numbers.
Section 2 gives sum* number and mod sum* number of $K_n - E(K_r)$. Section 3 gives sum* number and mod sum* number of the graphs $K_{n,n}$ and, $K_{n,n} - E(nK_2)$. Section 4

gives sum* number and mod sum* number of the graphs $P_n \odot K_1$ for $n \geq 2$ and $C_n \odot K_1$.

II. THE SUM*NUMBER AND MOD SUM* NUMBER OF $K_n - E(K_r)$.

In this section we determine the sum* number and mod sum* number of the graph $K_n - E(K_r)$. Let $G = K_n - E(K_r), n \geq r \geq 2$ and $m = \sigma * (G)$. We assume that $V(G \cup mK_1)$ is given sum* labeling so that we may denote the vertices of $G \cup mK_1$by their labels. Let $S = G \cup mK_1 = V\left(\left(K_n - E(K_r)\right) \cup mK_1\right)$; $A = V(K_r) = \{a_1, a_2, \cdots, a_r\}$, where $a_1 < a_2 < \cdots < a_r$ and a_i is not adjacent to a_j with $\neq j$; $B = V(K_n) \backslash V(K_r) = \{b_1 = 0, b_2, \cdots, b_{n-r}\}$, where $0 = b_1 < b_2 < \cdots < b_{n-r}$ and b_i is adjacent to $b_j$with $i \neq j$; $C = V(mK_1) = \{c_1, c_2, \cdots, c_m\}$, where $0 = b_1 < b_2 < \cdots < b_{n-r} < a_1 < a_2 < \cdots < a_r < c_1 < c_2 < \cdots < c_m$. So $C \cap (A \cup B) = \phi$. Let $V\left(\left(K_n - E(K_r)\right) \cup mK_1\right) = S = A \cup B \cup C$. Le $A_0 = \{b_i + a_j | j = 1, 2, \cdots, r; i = 2, 3, \cdots, n - r\}$ and $B_0 = \{b_i + b_j | i, j = 2, 3, \cdots, n - r \text{ and } i \neq j\}$..Thus, $A_0 \subset S, B_0 \subset S$ and as such $A_0 \cup B_0 \subseteq C \subset S$.

Lemma 2.1. $\mathbf{0 \in S}$.

Proof. It is obvious that $0 \in B$ and as such $0 \in A \cup B \cup C = S$.

Lemma2.2. $\boldsymbol{\sigma} * \left(K_n - E(K_r)\right) = 1$ for $n = 4$ and $r = 2$.

Proof. We consider the following sum* labeling of the graph $(K_n - E(K_r)) \cup K_1$:

$b_i = (i-1)N_1, i = 1, 2;$ $a_j = jN_1 + N_2, j = 1, 2;$ $c_k = (k+2)N_1 + N_2, k = 1$, where $N_1$and N_2 are prime numbers with $5 \leq N_1 \leq N_2$. Obviously, the above labeling is a sum* labeling of $K_n - E(K_r)$ and $\sigma * \left(K_n - E(K_r)\right) = 1$ for $n = 4$ and $r = 2$.

Lemma2.3. $\boldsymbol{\sigma} * \left(K_n - E(K_r)\right) = 0$ for $K_r \subseteq K_n$ and $3 \leq r \leq n \leq 4$.

Proof. It is easy to verify, so from now on we assume that $n \geq 5$ and $r \geq 2$.

Lemma 2. 4. $\boldsymbol{\sigma} * \left(K_n - E(K_r)\right) = 0$ for $K_r \subseteq K_n$ and $r = n$ or $r = n - 1$.

Proof. It is obvious that for $r = n$, since $K_n - E(K_n) = nK_1$. For $r = n - 1$, $K_n - E(K_r)$ is a star, which is known to be sum* graph [1]. □

Lemma 2. 5. $\boldsymbol{\sigma} * \left(K_n - E(K_r)\right) = n - r - 1$ for $K_r \subseteq K_n$ and $2 \leq r \leq n - 2$ and $n \geq 5$.

Proof. Label of the vertices in B are $0, 1, 2, \cdots, n - r - 1$ such that $\lambda(b_1) < \lambda(b_2) < \cdots < \lambda(b_{n-r})$ and label of the vertices in A are $n - r, n - r + 1, \cdots n - 1$ such that $\lambda(a_1) < \lambda(a_2) < \cdots < \lambda(a_r)$. Let S_1 be the set of $n - 1$ distinct labels produced by the edges incident on the vertex $b_1$and let S_2 be the set of $n - r$ distinct labels produced by the edges incident on the vertex a_r. The largest label in S_1, namely $\lambda(b_1) + \lambda(a_1)$ is the same as the smallest label in S_2 so that there are at least $(n-1) + (n-r) - 1 = 2n - r - 2$ distinct edge sums in a sum* labeling of the graph $K_n - E(K_r)$. Since the smallest label in the graph $K_n - E(K_r)$ cannot be the edge sum of any edge, at most $n - 1$ of these edge sums can be labels of the graph $K_n - E(K_r)$ so that $\boldsymbol{\sigma} * \left(K_n - E(K_r)\right) = (2n - r - 2) - (n - 1) = n - r - 1$. Label the vertices in B and A are respectively $0, 1, 2, \cdots, n - r - 1$ and $n - r, n - r + 1, \cdots n - 1$and also the incidentals with $n, n + 1, \cdots, 2n - r - 2$. □

Theorem2.1.

$$* \left(K_n - E(K_r)\right) = \begin{cases} 0 & if \quad r = n, n-1 \\ n - r - 1 & if \quad 2 \leq r \leq n - 2 \end{cases}$$ for $K_r \subseteq K_n$ and $n \geq 5$.

Proof. There are two valid assertions in this theorem. The first assertion is obtained from Lemma 2.4 and the second assertion is obtained from Lemma 2.5. □

Lemma 2.6. $\boldsymbol{\rho} * \left(K_n - E(K_r)\right) = 0$ for $K_r \subseteq K_n$ and $r = n$ or $r = n - 1$.

Proof. It is obvious that for $r = n$, since $K_n - E(K_n) = nK_1$. For $r = n - 1$, $K_n - E(K_r)$ is a star, which is known to be mod sum* graph [1]. □

Lemma2.7. $\boldsymbol{\rho} * \left(K_n - E(K_r)\right) = 0$ for $K_r \subseteq K_n$ and $2 \leq r \leq n - 2$ and $n \geq 5$.

Proof. Label of the vertices in B and A are respectively $0, 1, 2, \cdots, n - r - 1$ and $n - r, n - r + 1, \cdots n - 1$ and let graph modulus be $z = n$. Then all the edge sums (mod z) are vertices of the graph $K_n - E(K_r)$. So mod sum* number of the graph $K_n - E(K_r)$ is zero. Hence the Lemma 2.7 holds. □

Theorem 2.2. $\boldsymbol{\rho} * \left(K_n - E(K_r)\right) = 0$ for $K_r \subseteq K_n$ and $n \geq 5$.

Proof. There are two valid assertions in this theorem. The first assertion is obtained from Lemma 2.6 and the second assertion is obtained from Lemma 2.7. □

III. THE SUM*NUMBER AND MOD SUM* NUMBER OF THE GRAPHS $K_{n,n}$ AND $K_{n,n} - E(nK_2)$

The following open problems given by Sutton in his Ph.D. thesis [1] are solved in this section.

Open Problem 1. What is the sum* number and mod sum* number of the graph $K_{n,n}$.

Open Problem 2. What is the sum* number and mod sum* number of the graph $K_{n,n} - E(nK_2)$.

Theorem 3. 1. $\sigma * \left(K_{n,n}\right) = n - 1$ for $n \geq 2$.

Proof. The following facts are needed to prove the above theorem.

Fact 1: Let $m = \sigma*(K_{n,n}), n \geq 2$. Let $V(K_{n,n}) = (A,B)$ be the bipartition of a complete symmetric bipartite graph $K_{n,n}$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and a_i is not adjacent to a_j with $i \neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$, $C = V(mK_1) = \{c_1, c_2, \cdots, c_m\}$ be the set of incidentals, where $c_1 < c_2 < \cdots < c_m$. Hence, we have $(A \cup B) \cap C = \phi$. Let $V(K_{n,n} \cup mK_1) = A \cup B \cup C = S$.

Fact 2: Sum* labeling schemes for the graph $K_{n,n} \cup mK_1$ are given as follows.

$a_i = (i-1)N$, for $i = 1, 2, \cdots n$; $b_j = (j-1)N + 1$, for $j = 1, 2, \cdots n$; $c_k = (n+k-1)N + 1$, for $k = 1, 2, \cdots n-1$, where $N \geq 5$ is an integer.

It is obvious that $\{a_2 + b_n,\ a_3 + b_n,\ \cdots, a_n + b_n\} \subseteq C$ and $a_2 + b_n < a_3 + b_n < \cdots < a_n + b_n$. So $|C| = n-1$.

Fact 3:

- The vertices of S are distinct.
- $A \cap B = \emptyset, B \cap C = \emptyset, C \cap A = \emptyset$;
- $a_i + a_j \notin S$ for any $a_i, a_j \in A$ $(i \neq j \neq 1)$;
- $b_i + b_j \notin S$ for any $b_i, b_j \in A$ $(i \neq j)$;

Fact 4:

- $c_i + c_j \notin S$ for any $c_i, c_j \in C$ $(i \neq j)$;
- $a_i + c_j \notin S$ for any $a_i \in A$ $(i \neq 1)$ and for any $c_j \in C$;
- $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
- $a_i + b_j = c_k$ for any $a_i \in A$ $(i \neq 1)$ and for any $b_j \in B$.

Consequently from Fact 1 to 4 given above, we conclude that the above labeling is a sum* labeling of the graph $K_{n,n} \cup mK_1$ and as such the theorem holds. □

Theorem 3. 2. $\rho*(K_{n,n}) = 0$ for $n \geq 2$.

Proof. Consider the following facts.

Fact1: Let $V(K_{n,n}) = (A,B)$ be the bipartition of a complete symmetric bipartite graph $K_{n,n}$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ with a_i not adjacent to a_j with $i \neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$, Let $S = V(K_{n,n}) = A \cup B$.

Fact 2: Mod sum* labeling schemes for the graph $K_{n,n}$ as follows.

- $a_i = (i-1)N$, for $i = 1, 2, \cdots n$; $b_j = (j-1)N + 1$, for $j = 1, 2, \cdots n$; , where $N \geq 5$ is an integer with modulus $z = m = (n-1)N$.

Fact 3:

- The vertices of S are distinct.
- $A \cap B = \emptyset$;
- $a_i + a_j \notin S$ for any $a_i, a_j \in A$ $(i \neq j \neq 1)$;

Fact 4:

- $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j)$;
- $(a_i + b_j)(mod\ z)$ are vertices of $K_{n,n}$.

Facts 1 to 4 given above makes us conclude the labeling given above is a mod sum* labeling of graph $K_{n,n}$ and as such the proof follows. □

Theorem 3. 3. $\sigma*(K_{n,n} - E(nK_2)) = n - 3$ for $n \geq 4$.

Proof. Consider the following facts.

Fact 1: Let $m = \sigma*(K_{n,n} - E(nK_2)), n \geq 4$. Let $V(K_{n,n} - E(nK_2)) = (A,B)$ be the bipartition of $K_{n,n} - E(nK_2)$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and a_i is not adjacent to a_j with $i \neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$, $\{a_1b_1, a_2b_2, \cdots, a_nb_n\} = E(nK_2), C = V(mK_1) = \{c_1, c_2, \cdots, c_m\}$ be the set of incidentals, where $c_1 < c_2 < \cdots < c_m$. Hence, we have $(A \cup B) \cap C = \phi$. Let $S = V\left(\left(K_{n,n} - E(nK_2)\right) \cup mK_1\right) = A \cup B \cup C$.

Fact 2: Sum* labeling schemes for the graph $\left(K_{n,n} - E(nK_2)\right) \cup mK_1$ are given as follows.

$a_i = (i-1)N$, for $i = 1, 2, \cdots n$; $b_j = (j-1)N + 1$, for $j = 1, 2, \cdots n$; $c_k = (n+k-1)N + 1$, for $k = 1, 2, \cdots n-3$, where $N \geq 5$ is an integer.

It is obvious that $\{a_3 + b_n,\ a_4 + b_n,\ \cdots, a_{n-1} + b_n\} \subseteq C$ and $a_3 + b_n < a_4 + b_n < \cdots < a_{n-1} + b_n$. So $|C| = n-3$.

Fact 3:

- The vertices of S are distinct.
- $A \cap B = \emptyset, B \cap C = \emptyset, C \cap A = \emptyset$;
- $a_i + a_j \notin S$ for any $a_i, a_j \in A$ $(i \neq j \neq 1)$;
- $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j)$.

Fact 4:

- $c_i + c_j \notin S$ for any $c_i, c_j \in C$ $(i \neq j)$;
- $a_i + c_j \notin S$ for any $a_i \in A$ $(i \neq 1)$ and for any $c_j \in C$;
- $b_i + c_j \notin S$ for any $b_i \in B$ and for any $c_j \in C$;
- $a_i + b_j = c_k$ for any $a_i \in A$ $(i \neq 1)$ and for any $b_j \in B$;
- $a_i + b_j \notin S$ $(i \neq 1)$ iff $i + j = n + 2$ or $i = j = n$;
- $a_2 + b_n, a_3 + b_{n-1}, \cdots, a_n + b_2, a_n + b_n$ is $E(nK_2)$.

Consequently from Fact 1 to 4 given above, we conclude that the above labeling is a sum* labeling of the graph $\left(K_{n,n} - E(nK_2)\right) \cup mK_1$ and as such the theorem holds. □

Theorem 3.4. . $\rho*(K_{n,n} - E(nK_2)) = 0$ for $n \geq 4$.

Proof. Let $V(K_{n,n} - E(nK_2)) = (A,B)$ Let be the bipartition of the graph $(K_{n,n} - E(nK_2))$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and a_i is not adjacent to a_j with $i \neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$, $\{a_1b_1, a_2b_2, \cdots, a_nb_n\} = E(nK_2)$. Let Let $S = V\left(K_{n,n} - E(nK_2)\right) = A \cup B$. The mod sum* labeling for the graph $K_{n,n} - E(nK_2)$ is as follows.

$a_i = (i-1)N$, for $i = 1, 2, \cdots n$; $b_j = (j-1)N + 1$, for $j = 1, 2, \cdots n$; where $N \geq 5$ is an integer with modulus $z = m = (n-1)N$.

The following assertions are readily seen to be valid.

1. The vertices of $S$are distinct;
2. $A \cap B = \emptyset$;
3. $a_i + a_j \notin S$ for any $a_i, a_j \in A$ $(i \neq j \neq 1)$;
4. $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j)$;
5. $(a_i + b_j)(mod\ z)$ are vertices of $K_{n,n} - E(nK_2)$;
6. $a_i + b_j \notin S$ $(i \neq 1)$ iff $i + j = n + 2$ or $i = j = n$; So, $a_2 + b_n, a_3 + b_{n-1}, \cdots, a_n + b_2, a_n + b_n$ is $E(nK_2)$.

Thus the above labeling is a mod sum* labeling of graph $K_{n,n} - E(nK_2)$ and as such the theorem holds. □

IV. The sum*number and mod sum* number of the graphs $P_n \odot K_1$ and $C_n \odot K_1$

In this section, we determine sum* number and mod sum* number of the graphs $P_n \odot K_1$ and $C_n \odot K_1$. Recall that the sum* number is the minimum number of incidentals needed so that the union of graph and incidentals may be labeled as a sum* graph. Similarly the mod sum* number is the minimum number of incidentals needed so that the union of graph and incidentals may be labeled as a mod sum* graph.

Theorem 4.1. $\boldsymbol{\sigma} * (P_n \odot K_1) = \mathbf{0}$ for $n \geq 2$.

Proof. Let $V(P_n \odot K_1) = (A, B)$ be the bipartition of a graph $P_n \odot K_1$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and a_i is not adjacent to a_j with $i \neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$. Let $S = A \cup B$. The sum* labeling for the graph $P_n \odot K_1$ is as follows.

$\chi\{a_i\} = (i-1)$ for $i = 1, 2, \cdots, n$; $\chi\{b_j\} = 2n - j$ for $j = 1, 2, \cdots, n$.

Hence, we get $\chi\{a_i\} + \chi\{a_{i+1}\} \in S$ for $i = 1, 2, \cdots, n-1$ and $\chi\{a_i\} + \chi\{b_i\} = 2n - 1$ for $i = 1, 2, \cdots, n$.

The above labeling is a sum* labeling of the graph $P_n \odot K_1$ *and* $\boldsymbol{\sigma} * (P_n \odot K_1) = \mathbf{0}$ for $n \geq 2$. □

Theorem 4.2. $\boldsymbol{\rho} * (P_n \odot K_1) = \mathbf{0}$ for $n \geq 2$.

Proof. Let $V(P_n \odot K_1) = (A, B)$ be the bipartition of a graph $P_n \odot K_1$ with $A = \{a_1, a_2, \cdots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$ and a_i is not adjacent to a_j with $\neq j$, $B = \{b_1, b_2, \cdots, b_n\}$, where $b_1 < b_2 < \cdots < b_n$ and b_i is not adjacent to b_j with $i \neq j$. Let $S = A \cup B$. The mod sum* labeling for the graph $P_n \odot K_1$ is as follows.

$\chi\{a_i\} = (i-1)$ for $i = 1, 2, \cdots, n$; $\chi\{b_j\} = 2n - j$ for $j = 1, 2, \cdots, n$ with modulus $z = 2n$.

Thus the above labeling is a mod sum* labeling of the graph $P_n \odot K_1$ and $*(P_n \odot K_1) = \mathbf{0}$ for $n \geq 2$. □

Theorem 4.3. . $\boldsymbol{\sigma} * (C_n \odot K_1) = \mathbf{0}$ for $n \geq 4$.

Proof. We have either $n = 2k + 1$ $(k \geq 2)$ or $n = 2k$ $(k \geq 2)$.

Case 1: Let $n = 2k + 1$ $(k \geq 2)$.. Consider the following cases.

Fact 1: Sum* labeling schemes for the graph $C_n \odot K_1$ as follows.

- $a_i = (k-i)N + 1$, $b_i = (i-1)N$, $i = 1, 2, \cdots, k$;
- $d_i = (2k + i - 2)N + 2$, $e_i = (3k - i - 1)N + 3$, $i = 1, 2, \cdots, k$;
- $a_{k+1} = kN + 1$, $d_{k+1} = kN + 2$, where $N \geq 5$ is an integer;
- Let $A = \{a_1, a_2, \cdots, a_k, a_{k+1}\}$, $B = \{b_1, b_2, \cdots, b_k\}$, $D = \{d_1, d_2, \cdots, d_k, d_{k+1}\}$, $E = \{e_1, e_2, \cdots, e_k\}$, $S = A \cup B \cup D \cup E$.

Fact 2:

- The vertices of S are distinct;
- $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j, i \neq 1)$;
- $d + d_j \notin S$ for any $d_i, d_j \in D$ $(i \neq j)$;
- $e_i + e_j \notin S$ for any $e_i, e_j \in E$ $(i \neq j)$.

Fact 3:

- $a_i + b_j \in S$ if and only if a_i is adjacent to b_j;
- $a_i + d_j \in S$ if and only if $i = j$ and $a_1b_ka_2b_{k-1} \cdots a_{k-1}b_2a_kb_1a_{k+1}a_1$ is a cycle C_{2k+1};
- $a_i + e_j \notin S$ for any $a_i \in A$ and for any any $e_j \in E$;
- $b_i + d_j \notin S$ for any $b_i \in B$ and for any any $d_j \in D$;
- $b_i + e_j \in S$ if and only if $i = j$.

Case 2: Let $n = 2k$ $(k \geq 2)$. Consider the following facts.

Facts 1: Sum* labeling schemes for the graph $C_n \odot K_1$ is as follows.

- $a_i = (k-i)N + 1$, $b_i = (i-1)N$, $i = 1, 2, \cdots, k$.
- $d_i = (k + i - 1)N + 1$, $e_i = (3k - i - 1)N + 2$, $i = 1, 2, \cdots, k$, where $N \geq 5$ is an integer.
- Let $A = \{a_1, a_2, \cdots, a_k\}$, $B = \{b_1, b_2, \cdots, b_k\}$, $D = \{d_1, d_2, \cdots, d_k\}$, $E = \{e_1, e_2, \cdots, e_k\}$, $S = A \cup B \cup D \cup E$.

Fact 2:

- The vertices of S are distinct;
- $b_i + b_j \notin S$ for any $b_i, b_j \in B$ $(i \neq j, i \neq 1)$;
- $d_i + d_j \notin S$ for any $d_i, d_j \in D$ $(i \neq j)$;
- $e_i + e_j \notin S$ for any $e_i, e_j \in E$ $(i \neq j)$.
- $a_i + b_j \in S$ iff a_i is adjacent to b_j.

Fact 3:

- $a_i + d_j \in S$ if and only if $i = j$ and $a_1 b_k a_2 b_{k-1} \cdots a_{k-1} b_2 a_k b_1 a_1$ is a cycle C_{2k};
- $a_i + e_j \notin S$ for any $a_i \in A$ and for any any $e_j \in E$;
- $b_i + d_j \notin S$ for any $b_i \in B$ and for any any $d_j \in D$;
- $b_i + e_j \in S$ if and only if $i = j$;
- $d_i + e_j \notin S$ for any $d_i \in D$ and for any any $e_j \in E$.

From the facts given above, we conclude that the above labeling is a sum* labeling of graph $C_n \odot K_1$ for $n \geq 4$ with $\boldsymbol{\sigma} * (C_n \odot K_1) = \mathbf{0}$ and hence the proof. □

Theorem 4.4. . $\boldsymbol{\rho} * (C_n \odot K_1) = \mathbf{0}$ for $n \geq 4$.

Proof. We have either $n = 2k + 1\ (k \geq 2)$ or $n = 2k\ (k \geq 2)$.

Case 1: Let $n = 2k + 1\ (k \geq 2)$.. Consider the following cases.

Fact 1: Mod sum* labeling scheme for the graph $C_n \odot K_1$ is as follows.

- $a_i = (k - i)N + 1,\ b_i = (i - 1)N, i = 1, 2, \cdots, k$;
- $d_i = (k - i)N + 2,\ e_i = (k - i)N + 3, i = 1, 2, \cdots, k$;
- $a_{k+1} = kN + 1, d_{k+1} = kN + 2$ with modulus $z = kn$, where $N \geq 5$ is an integer;
- Let $A = \{a_1, a_2, \cdots, a_k, a_{k+1}\}, B = \{b_1, b_2, \cdots, b_k\}, D = \{d_1, d_2, \cdots, d_k, d_{k+1}\}, E = \{e_1, e_2, \cdots, e_k\}, S = A \cup B \cup D \cup E$.

Fact 2:

- The vertices of S are distinct;
- $a_i + b_j \bmod z \in S$ if and only if a_i is adjacent to b_j;[1.]
- $a_i + d_j \bmod z \in S$ if and only if $i = j$;
- $b_i + e_j \bmod z \in S$ if and only if $i = j$;
- $a_1 b_k a_2 b_{k-1} \cdots a_{k-1} b_2 a_k b_1 a_{k+1} a_1$ is a cycle C_{2k+1}.

Case 2: Let$n = 2k\ (k \geq 2)$. Consider the following facts.

Fact1: Mod sum* labeling scheme for the graph $C_n \odot K_1$ is as follows.

- $a_i = (k - i)N + 1,\ b_i = (i - 1)N, i = 1, 2, \cdots, k$;
- $d_i = (k + i - 1)N + 1,\ e_i = (3k - i - 1)N + 2, i = 1, 2, \cdots, k$ with modulus $z = (n + k - 1)N$, where $N \geq 5$ is an integer;
- Let $A = \{a_1, a_2, \cdots, a_k\}, B = \{b_1, b_2, \cdots, b_k\}, D = \{d_1, d_2, \cdots, d_k\}, E = \{e_1, e_2, \cdots, e_k\}, S = A \cup B \cup D \cup E$.

Fact 2:

- The vertices of S are distinct;
- $a_i + b_j \bmod z \in S$ if and only if a_i is adjacent to b_j;
- $a_i + d_j \bmod z \in S$ if and only if $i = j$;
- $b_i + e_j \bmod z \in S$ if and only if $i = j$;
- $a_1 b_k a_2 b_{k-1} \cdots a_{k-1} b_2 a_k b_1 a_1$ is a cycle C_{2k};
- $d_i + e_j \notin S$ for any $d_i \in D$ and for any any $e_j \in E$.

From the facts given above, we conclude that the above labeling is a mod sum* labeling of graph $C_n \odot K_1$ for $n \geq 4$ with $\rho * (C_n \odot K_1) = \mathbf{0}$ and hence the proof. □

V. CONCLUSION

This article gives sum* and mod sum* labeling to some new classes of graphs and determine their sum* and mod sum* numbers. The graphs explored in this article include $K_n - E(K_r)$, $K_{n,n}$ for $n \geq 2$, $K_{n,n} - E(nK_2)$ for $n \geq 2$, $P_n \odot K_1$ for $n \geq 2$ and $C_n \odot K_1$ for $n \geq 4$. To the best of our knowledge this is the second article dealing with sum* and mod sum* labeling and the only other article available in literature is the one given by M. Sutton [1] who introduced these labeling concepts. As not much progress has been done in this area there is huge scope for further explorations.

REFERENCES

[1] M. Sutton, "Sumable Graphs Labellings and Their Applications", Ph. D. Thesis, Department of Computer Science, The University of Newcastle, 2001.

[2] F.Harary, "Graph Theory", Addison-Wesley, Reading, MA, 1969.

[3] F. Harary, "Sum Graphs and Difference Graphs", Congressus Numerantium , Vol. 72 , pp. 101 - 108, 1990.

[4] J. Bolland, R. Laskar, C. Turner, G. Domke, " On Mod Sum Graphs" , Congressus Numerantium, Vol. 70, pp.131-135, 1990.

[5] W. Dou, J. Gao, "The (Mod, Integral) Sum Numbers of Fans and $K_{n,n} - E(nK_2)$", Discrete Mathematics, Vol. 306, pp.2655-2669, 2006.

[6] H.Wang, P. Li, "Some Results on Exclusive Sum Graphs", J. Appl. Math Comput, Vol. 34, pp. 343-351, 2010.

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