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Finite Element Method to Study Elasto-Thermodiffusive Response inside a Hollow Cylinder with Three-Phase-Lag Effect

Debarghya Bhattacharya[1*], Prashanta Pal[2], Mridula Kanoria[3]

[1,2,3]Dept. of Applied Mathematics, University of Calcutta, 92 A. P. C. Road, Kolkata- 700009, West Bengal, India

[*]*Corresponding Author: debbhattacharya88@gmail.com Tel.: +918961328903*

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Abstract— In this article, we deal with a problem of generalized elasto-thermo-diffusion interaction inside an isotropic hollow cylinder in the context of three-phase-lag model. Initially the medium is in rest and undisturbed so that all the state functions are assumed to be zero. Employing Laplace transform as a tool, the governing equations have been expressed in transformed domain, which are then solved by Galerkin finite element technique. The inversion of the transformed solution is carried out by applying a method of Bellman et al. The stresses, temperature, displacement, concentration and chemical potential are computed numerically and presented graphically in a number of figures for copper material. A comparative study for different theories (three-phase-lag model, Green-Naghdi model with energy dissipation and Lord-Shulman model) are presented. The results corresponding to thermoelastic medium (in absence of diffusion) are also carried out in a particular case. The significant points are highlighted.

Keywords—Thermoelastic diffusion, Three-phase-lag model, Green-Naghdi model, Lord-Shulman model, Galerkin finite element method.

LIST OF SYMBOLS

$\bar{u}$	Displacement vector
θ	Thermodynamic temperature
λ, μ	Lame's constants
ρ	Constant mass density of the medium
T	Temperature
T_0	Uniform reference temperature
C	Mass concentration
K	Thermal conductivity
D	Diffusion coefficient
c_E	Specific heat
P	Chemical potential
σ_{ij}	Components of stress tensor
c, d	Measures of thermo-diffusion effect and diffusion effect
β_1, β_2	Material constants

I. INTRODUCTION

The topic 'thermoelastic diffusion' also termed as 'elasto-thermo-diffusion' in an elastic solid is one of the transport process that have extensive applications in the field of geophysics, in the extraction of oil from oil deposits and other industrial applications. The thermodiffusion process also helps the investigation in the field associated with the advancement of microelectronics, many problem of satellites, returning space vehicles, and landing on water or land.

For the first time, the theory of thermoelastic diffusion using the coupled thermoelastic model was developed by Nowacki [1-4]. In this coupled theory the heat propagation is assumed to be infinitely large velocity. This contradicts physical observations. In order to overcome the paradox of infinite speed of thermal wave inherent in classical coupled thermoelasticity theory, the subject 'Generalized Thermoelasticity' is developed. Lord and Shulman [5] formulated the generalized thermoelasticity theory. In which the conventional Fourier's law is replaced by a modified law of heat conduction including both the heat flux and its time derivative. This theory is referred to as extended thermoelasticity theory (ETE) or L-S theory. The heat equation associated with this a hyperbolic one and hence, automatically eliminates the paradox of infinite speeds of propagation inherent in the coupled theory of thermoelasticity.

Sherief et al. [6] investigated the theory of generalized thermoelastic diffusion with one relaxation time, which allow the finite speeds of propagation of waves inside the medium. Sherief and Salah [7] investigated the problem of a thermoelastic half space in the context of generalized thermoelastic diffusion with one relaxation time. Aouadi [8] proved the uniqueness and reciprocity theorems for the equations of generalized thermoelastic diffusion problem, in an isotropic media using Laplace transformation method. He also studied the effect of diffusion in an infinitely long solid cylinder [9] and in an infinite elastic body with spherical cavity [10]. Problem related to two-temperature thermoelastic diffusion have been investigated by Bhattacharya and Kanoria [11]. Recently problems related to generalized thermoelasticity under the effect of diffusion have been discussed by many researchers [12-14].

Green-Naghdi [15-17] developed three models for generalized thermoelasticity of homogeneous isotropic material which is labeled as I, II, III. An important feature of GN-III theory is that it accommodates dissipation of thermal energy due to the presence of thermal damping term. A theorem on uniqueness of solutions in the context of a linearized version of this theory has been established by Chandrasekharish [18].

The most relevant development in thermoelasticity theory is three-phase-lag model which was developed by Roychoudhuri [21]. In this model the Fourier's law of heat conduction is replaced by an approximation to a modified form with the introduction of different phase lags for the heat flux vector, temperature gradient and for the thermal displacement gradient. According to this model, $\vec{q}(P,t+\tau_q) = -\left[K\vec{\nabla}\theta(P,t+\tau_T) + K^*\vec{\nabla}\nu(P,t+\tau_\nu)\right]$, where $\vec{\nabla}\theta$ is the temperature gradient at a point P of the material at time $t+\tau_T$, $\vec{q}$ is the heat flux vector at the point P of the material at time $t+\tau_q$, $\vec{\nu}$ $(\dot{\nu} \equiv \theta)$ is the thermal displacement gradient, K^* is the additional material constant and K is the thermal conductivity of the material. Three-phase-lag model is very useful in the problems of nuclear bonding, exothermic catalytic reactions, photon-electron interactions, photon-scattering. Quintanilla and Racke [22] have discussed the stability of solutions in three-phase-lag effect. Kar and Kanoria [23] have studied thermoelastic response in a fiber reinforced thin annular disc with three-phase-lag effect. Kanoria et al. [24] dealt with the problem of magneto-thermoelastic response in a transversely isotropic hollow cylinder under thermal shock with three-phase-lag effect. Pal and Kanoria [25] applied finite element method to study magneto-thermoelastic wave in a transversely isotropic hollow cylinder under three phase-lag model. Problem related to two-temperature elasto-thermodiffusive response in an isotropic shell with three-phase-lag model has been solved by Kanoria et al. [26]. Some qualitative analysis using three-phase-lag model have been established by Said [27], Biswas et al. [28].

The main object of the present contribution is to present thermo-diffusive interaction in an isotropic hollow cylinder under three-phase-lag model, Green-Naghdi III (GN-III) model and Lord-Shulman model. The governing equations of the theory of thermoelastic diffusion are obtained in Laplace transform domain which are then solved by Galerkin finite element method. The inversion of the transformed solution is carried out by applying a method of Bellman et al.[29]. A complete and comprehensive analysis of the results have been presented for three-phase-lag model, GN-III and Lord-Shulman model in presence of diffusion (WD) as well as in absence of diffusion (WOD). The significant points arising out from our analysis are highlighted.

II. FORMULATION OF THE PROBLEM:

We shall consider a homogeneous isotropic thermoelastic hollow cylinder of inner radius a and outer radius b in an undisturbed state at initial temperature T_0. We use the cylindrical coordinate system (r,θ,z) with $z-$axis coincident with the axis of cylinder. Due to axial symmetry, the problem is one dimensional with all the considered functions depending on the radial distance r and the time t. The outer surface of the cylinder is taken to be traction free and is subjected to a time dependent thermal shock and chemical shocks.

If $\bar{u} = [u(r,t),0,0]$ be the displacement vector, then the strain vector e has components $e_{rr} = \dfrac{\partial u}{\partial r}$, $e_{\theta\theta} = \dfrac{u}{r}$, $e_{zz} = e_{rz} = e_{r\theta} = e_{\theta z} = 0.$

Therefore, basic equation for the problem is considered as

$$\sigma_{ij} = 2\mu e_{ij} + [\lambda e - \beta_1 T - \beta_2 C]\delta_{ij}, \qquad i,j = 1,2,3. \tag{1}$$

In cylindrical coordinates the equation of motion in the absence of body forces is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho \frac{\partial^2 u}{\partial t^2}. \tag{2}$$

Now, the generalized heat conduction equation in the context of three-phase-lag model is

$$K^*\nabla^2 T + \tau_\upsilon^* \nabla^2 \dot{T} + K\tau_T \nabla^2 \ddot{T} = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\left(\rho c_E \ddot{T} + T_0 \beta_1 \ddot{e} + cT_0 \ddot{C}\right). \tag{3}$$

The equation of mass diffusion is

$$D\beta_2 \nabla^2 e + Dc\nabla^2 T + \dot{C} + \tau^0 \ddot{C} = Dd\nabla^2 C. \tag{4}$$

The chemical potential is given by,

$$P = -\beta_2 e + dC - cT. \tag{5}$$

Due to cylindrical symmetry, the stress-strain-temperature-concentration relations take the form

$$\sigma_{rr} = (\lambda+2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r} - \beta_1 T - \beta_2 C, \quad (6)$$

$$\sigma_{\theta\theta} = (\lambda+2\mu)\frac{u}{r} + \lambda\frac{\partial u}{\partial r} - \beta_1 T - \beta_2 C, \quad (7)$$

where K^* is the additional material constant, τ_T and τ_q are the phase lag of temperature gradient and the phase-lag of heat flux respectively. τ^0 is the diffusion relaxation time; C is the mass concentration such that $\eta_{i,i} = -\dot{C}$; η_j is the flow of the diffusing mass vector.

Also $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

and $\tau_\upsilon^* = K + \tau_\upsilon K^*$, where τ_υ is the phase lag of thermal displacement gradient.

$\beta_1 = (3\lambda+2\mu)\alpha_t$, $\beta_2 = (3\lambda+2\mu)\alpha_c$; α_t is the coefficient of linear thermal expansion of the material, α_c is the coefficient of linear diffusion expansion.

Using (6) and (7), the equation of motion has the form:

$$(\lambda+2\mu)\frac{\partial e}{\partial r} - \beta_1\frac{\partial T}{\partial r} - \beta_2\frac{\partial C}{\partial r} = \rho\ddot{u}. \quad (8)$$

Special cases:

(1) When $\tau_T = 0$, $\tau_q = 0$ and $\tau_\upsilon = 0$, then $\tau_\upsilon^* = K$. Thus Eq. (3) becomes

$$K^*\nabla^2 T + \tau_\upsilon^*\nabla^2\dot{T} = (\rho c_E\ddot{T} + T_0\beta_1\ddot{e} + cT_0\ddot{C}),$$

which is third model of Green-Naghdi admitting damped thermoelastic wave equation with diffusion.

(2) When $\tau_T = 0 = \tau_q^2$, $K^* = 0$, $\tau_\upsilon = 0$, this theory reduce to L-S theory.

Now it is convenient to change the preceding equations into the dimensionless form. To do this, the dimensionless parameters are introduced as,

$$R' = \frac{r}{a}, S' = \frac{b}{a}, t' = \frac{Gt}{a}, \theta' = \frac{T}{T_0}, (\tau_q', \tau_T', \tau_\upsilon', \tau'^0) = \frac{G}{a}(\tau_q, \tau_T, \tau_\upsilon, \tau^0),$$

$$U' = \frac{(\lambda+2\mu)}{a\beta_1 T_0}u, G^2 = \frac{(\lambda+2\mu)}{\rho}, C' = \frac{\beta_2}{\beta_1 T_0}C, (\sigma_R, \sigma_\theta) = \frac{1}{\beta_1 T_0}(\sigma_{rr}, \sigma_{\theta\theta}).$$

Using these non-dimensional variables and omitting primes, Eqs. (8), (3), (4), (6), (7) and (5) become

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{U}{R^2} = \frac{\partial^2 U}{\partial t^2} + \frac{\partial\theta}{\partial R} + \frac{\partial C}{\partial R}, \quad (9)$$

$$\left[C_T^2 + (C_K^2 + \tau_\upsilon C_T^2)\frac{\partial}{\partial t} + \tau_T C_K^2\frac{\partial^2}{\partial t^2}\right]\nabla^2\theta = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left(\ddot{\theta} + \varepsilon\ddot{e} + \varepsilon\alpha_1\ddot{C}\right), \quad (10)$$

$$\nabla^2 e + \alpha_1\nabla^2\theta + \dot{C} + \alpha_2(\dot{C} + \tau^0\ddot{C}) = \alpha_3\nabla^2 C, \quad (11)$$

$$\sigma_R = \frac{\partial U}{\partial R} + c_2\frac{U}{R} - \theta - C, \quad (12)$$

$$\sigma_\theta = \frac{U}{R} + c_2\frac{\partial U}{\partial R} - \theta - C, \quad (13)$$

$$P = -e + \alpha_3 C - \alpha_1\theta, \quad (14)$$

where

$$C_T^2 = \frac{K^*}{\rho c_E G^2}, C_K^2 = \frac{K}{a\rho c_E G^2}, \alpha_1 = \frac{c(\lambda+2\mu)}{\beta_1\beta_2}, \alpha_2 = \frac{\rho G(\lambda+2\mu)}{D\beta_2^2},$$

$$\alpha_3 = \frac{d\rho(\lambda+2\mu)}{\beta_2^2}, \varepsilon = \frac{T_0\beta_1^2}{\rho c_E(\lambda+2\mu)}, c_2 = \frac{\lambda}{(\lambda+2\mu)}.$$

The mechanical, thermal and chemical boundary conditions are given by,

$\sigma_R = 0$ on $R = 1, S$ (dimensionless)

$\theta = 0$ on $R = 1;\ t > 0,$

$\frac{\partial\theta}{\partial R} = \theta_b H(t)$ on $R = S;\ t > 0,$

$P = 0$ on $R = 1;\ t > 0,$

$\frac{\partial P}{\partial R} = P_b H(t)$ on $R = S;\ t > 0.$

All the state functions are assumed to be zero, as the medium initially is at rest and undisturbed. Here the terms $\theta_b H(t)$ and $P_b H(t)$ are heat flux and mass flux applied to the outer boundary of the cylinder $R = S$, where $H(t)$ is the Heaviside unit-step function.

III. METHOD OF SOLUTION

Let

$$\left[\bar{U}(R,s), \bar{\theta}(R,s), \bar{C}(R,s)\right] = \int_0^\infty \left[U(R,\zeta), \theta(R,\zeta), C(R,\zeta)\right]e^{-s\zeta}d\zeta, \quad \mathrm{Re}(s) > 0 \quad (15)$$

denote the Laplace transformation of U, θ and C respectively. On taking the Laplace transform, Eqs. (9)-(14) become:

$$\frac{\partial^2\bar{U}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{U}}{\partial R} - \frac{\bar{U}}{R^2} = s^2\bar{U} + \frac{\partial\bar{\theta}}{\partial R} + \frac{\partial\bar{C}}{\partial R}, \quad (16)$$

$$\frac{\partial^2\bar{\theta}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{\theta}}{\partial R} = a_3\left[\bar{\theta} + \varepsilon\left(\frac{\partial\bar{U}}{\partial R} + \frac{\bar{U}}{R}\right) + \varepsilon\alpha_1\bar{C}\right], \quad (17)$$

$$\frac{\partial^3\overline{U}}{\partial R^3}+\frac{2}{R}\frac{\partial^2\overline{U}}{\partial R^2}-\frac{1}{R^2}\frac{\partial\overline{U}}{\partial R}+\frac{\overline{U}}{R^3}+\alpha_1\left(\frac{\partial^2\overline{\theta}}{\partial R^2}+\frac{1}{R}\frac{\partial\overline{\theta}}{\partial R}\right)+ \tag{18}$$

$$\alpha_2 s(1+\tau^0 s)\overline{C}=\alpha_3\left(\frac{\partial^2\overline{C}}{\partial R^2}+\frac{1}{R}\frac{\partial\overline{C}}{\partial R}\right),$$

$$\overline{\sigma}_R=\frac{\partial\overline{U}}{\partial R}+c_2\frac{\overline{U}}{R}-\overline{\theta}-\overline{C}, \tag{19}$$

$$\overline{\sigma}_\theta=\frac{\overline{U}}{R}+c_2\frac{\partial\overline{U}}{\partial R}-\overline{\theta}-\overline{C}, \tag{20}$$

$$P=-\left(\frac{\partial\overline{U}}{\partial R}+\frac{\overline{U}}{R}\right)+\alpha_3\overline{C}-\alpha_1\overline{\theta}, \tag{21}$$

where $a_3=\dfrac{\left(1+\tau_q s+\frac{1}{2}\tau_q^2 s^2\right)}{\left(C_T^2+(C_K^2+\tau_\upsilon C_T^2)s+\tau_T C_K^2 s^2\right)}.$

IV. FINITE ELEMENT ANALYSIS

The Galerkin finite element method is used to derive the stiffness and force matrices for the base element (e). In this finite element method, total domain is divided into a finite set of sub-intervals, i.e. line elements along the radial direction of the disc. For any base element (e), the displacement $\overline{U}$, the temperature $\overline{\theta}$ and the concentration $\overline{C}$ can be approximated as

$$\overline{U}=[N]^{(e)}[U^*]^{(e)},\overline{\theta}=[N]^{(e)}[\theta^*]^{(e)},\overline{C}=[N]^{(e)}[C^*]^{(e)}, \tag{22}$$

where $[N]^{(e)}$ is the shape function approximating the displacement, temperature and concentration field in the base element (e). The matrices $[U^*]^{(e)},[\theta^*]^{(e)}$ and $[C^*]^{(e)}$ represent the nodal values of the displacement, temperature and concentration respectively. Using Eq. (22) and Galerkin finite element method over the volume of the base element $V^{(e)}$, Eqs. (16), (17) and (18) become

$$\int_{V^{(e)}}\left[\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}-\frac{1}{R^2}-s^2\right)\overline{U}-\frac{\partial\overline{\theta}}{\partial R}+\frac{\partial\overline{C}}{\partial R}\right]N_m dV=0, \tag{23}$$

$$\int_{V^{(e)}}\left[\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}-a_3\right)\overline{\theta}-a_3\varepsilon\left(\frac{d}{dR}+\frac{1}{R}\right)\overline{U}+\varepsilon\alpha_1 a_3\overline{C}\right]N_m dV=0, \tag{24}$$

$$\int_{V^{(e)}}\left[\left(\frac{d^3}{dR^3}+\frac{2}{R}\frac{d^2}{dR^2}-\frac{1}{R^2}\frac{d}{dR}+\frac{1}{R^3}\right)\overline{U}+\alpha_1\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}\right)\overline{\theta}+ \tag{25}$$

$$\left.\left\{\alpha_2 s(1+\tau^0 s)-\alpha_3\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}\right)\right\}\right]\overline{C}N_m dV=0,$$

where N_m is the shape function. Using the local coordinates $R^*=R-R_i$, where R_i is the radius of the i th node of the base element, Eqs. (23), (24) and (25) reduce to (dropping the asterisk for convenience):

$$\int_0^L\left[-\left\{\left(\frac{1}{(R+R_i)}\frac{d}{dR}-\frac{1}{(R+R_i)^2}-s^2\right)\overline{U}-\frac{d\overline{\theta}}{dR}-\frac{d\overline{C}}{dR}\right\}N_m(R+R_i)+ \tag{26}$$

$$\left.\frac{d}{dR}\{N_m(R+R_i)\}\frac{d\overline{U}}{dR}\right]dR=\{N_m(R+R_i)\}\frac{d\overline{U}}{dR}\Big|_0^L$$

$$\int_0^L\left[-\left\{\left(\frac{1}{(R+R_i)}\frac{d}{dR}-a_3\right)\overline{\theta}-\varepsilon a_3\left(\frac{d}{dR}+\frac{1}{(R+R_i)}\right)\overline{U}-\right.\right.$$

$$\left.\left.a_3\varepsilon\alpha_1\overline{C}\right\}N_m(R+R_i)+\frac{d}{dR}\{N_m(R+R_i)\}\frac{d\overline{\theta}}{dR}\right]dR= \tag{27}$$

$$\{N_m(R+R_i)\}\frac{d\overline{\theta}}{dR}\Big|_0^L,$$

$$\int_0^L\left[-\frac{d^2}{dR^2}\{N_m(R+R_i)\}\frac{d}{dR}+\frac{d}{dR}(2N_m)\frac{d}{dR}+\right.$$

$$N_m\left(\frac{1}{(R+R_i)}\frac{d}{dR}-\frac{1}{(R+R_i)^2}\right)U+\alpha_1\left\{\frac{d}{dR}\{N_m(R+R_i)\}\frac{d}{dR}-N_m\frac{d}{dR}\right\}\overline{\theta}+$$

$$\left.\alpha_3\{N_m\frac{d}{dR}-\frac{d}{dR}\{N_m(R+R_i)\}\frac{d}{dR}-\alpha_2 s(1+\tau^0 s)N_m(R+R_i)\}\overline{C}\right]dR=$$

$$\{N_m(R+R_i)\}\frac{d^2\overline{U}}{dR^2}\Big|_0^L+2N_m\frac{d\overline{U}}{dR}\Big|_0^L+\alpha_1\{N_m(R+R_i)\}\frac{d\overline{\theta}}{dR}\Big|_0^L-$$

$$\alpha_3\{N_m(R+R_i)\}\frac{d\overline{C}}{dR}\Big|_0^L-\frac{d}{dR}\{N_m(R+R_i)\}\frac{d\overline{U}}{dR}\Big|_0^L, \tag{28}$$

where $L=R_j-R_i$ is the length of the elements along the radial direction.

Now the applied boundary conditions may be considered as:

$$\overline{\theta}=0,\quad \text{on}\quad R=1, \tag{29}$$

$$b\frac{d\overline{\theta}}{dR}=b\frac{\theta_b}{s},\quad \text{on}\quad R=S, \tag{30}$$

$$\overline{P}=0,\quad \text{on}\quad R=1, \tag{31}$$

$$b\frac{d\overline{P}}{dR}=b\frac{P_b}{s},\quad \text{on}\quad R=S, \tag{32}$$

$$\frac{d\overline{U}}{dR}\Big|_1=\frac{(1-c_2\alpha_3)}{a(\alpha_3-1)}\overline{U}_1,\quad \text{on}\quad R=1, \tag{33}$$

$$\frac{d\overline{U}}{dR}\Big|_M+\frac{c_2}{b}\overline{U}\Big|_M-\overline{\theta}\Big|_M-\overline{C}\Big|_M=0, \tag{34}$$

where 1 and M denote the first and last nodes of the solution domain respectively.

Substituting the Eq. (22) into the Eqs (26)- (28) and after dropping the asterisks for convenience, we get

$$\int_0^L \overline{U}_n\left[-N_m\frac{dN_n}{dR}+\frac{N_mN_n}{(R+R_i)}+s^2N_mN_n(R+R_i)+\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}\right]dR$$
$$+\int_0^L \overline{\theta}_n\left[\frac{dN_n}{dR}N_m(R+R_i)\right]dR+\int_0^L \overline{C}_n\left[\frac{dN_n}{dR}N_m(R+R_i)\right]dR=$$
$$\{N_m(R+R_i)\}\frac{d\overline{U}_n}{dR}\Big|_0^L, \tag{35}$$

$$\int_0^L \overline{U}_n[\varepsilon a_3\{N_m(R+R_i)\frac{dN_n}{dR}+N_mN_n\}]dR+\int_0^L \overline{\theta}_n[\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-N_m\frac{dN_n}{dR}+$$
$$a_3N_mN_n(R+R_i)]dR+\int_0^L \overline{C}_n a_3\varepsilon\alpha_1N_mN_n(R+R_i)dR=\{N_m(R+R_i)\}\frac{d\theta}{dR}\Big|_0^L, \tag{36}$$

$$\int_0^L \overline{U}_n\left[-\frac{d^2}{dR^2}\{N_m(R+R_i)\}\frac{dN_n}{dR}+\frac{d}{dR}(2N_m)\frac{dN_n}{dR}+\left\{\frac{N_m}{(R+R_i)}\frac{dN_n}{dR}-\frac{N_mN_m}{(R+R_i)^2}\right\}\right]dR+$$
$$\int_0^L \overline{\theta}_n\alpha_1\left[\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-N_m\frac{dN_n}{dR}\right]dR+\int_0^L \overline{C}_n\alpha_3$$
$$\left[N_m\frac{dN_m}{dR}-\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-\alpha_2 s(1+\tau^0 s)N_mN_n(R+R_i)\right]dR$$
$$=\{N_m(R+R_i)\}\frac{d^2\overline{U}}{dR^2}\Big|_0^L+2N_m\frac{d\overline{U}}{dR}\Big|_0^L+$$
$$\alpha_1\{N_m(R+R_i)\}\frac{d\overline{\theta}}{dR}\Big|_0^L-\alpha_3\{N_m(R+R_i)\}\frac{d\overline{C}}{dR}\Big|_0^L-\frac{d}{dR}\{N_m(R+R_i)\}\frac{d\overline{U}}{dR}\Big|_0^L. \tag{37}$$

Now the transfinite element Eqs. (35)- (37) are expressed in the matrix form as

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\begin{bmatrix} \overline{U} \\ \overline{\theta} \\ \overline{C} \end{bmatrix}=\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{38}$$

where the sub-matrices $K_{11}, K_{12}, K_{13}, K_{21}, K_{22}, K_{23}, K_{31}, K_{32}, K_{33}, X, Y, Z$ are

$$K_{11}^{mn}=\int_0^L\left[-N_m\frac{dN_n}{dR}+\frac{N_mN_n}{(R+R_i)}+s^2N_mN_n(R+R_i)+\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}\right]dR, \tag{39}$$

$$K_{12}^{mn}=\int_0^L\left[\frac{dN_n}{dR}N_m(R+R_i)\right]dR, \tag{40}$$

$$K_{13}^{mn}=\int_0^L\left[\frac{dN_n}{dR}N_m(R+R_i)\right]dR, \tag{41}$$

$$K_{21}^{mn}=\int_0^L\left[\varepsilon a_3\{N_m(R+R_i)\frac{dN_n}{dR}+N_mN_n\}\right]dR, \tag{42}$$

$$K_{22}^{mn}=\int_0^L\left[\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-N_m\frac{dN_n}{dR}+a_3N_mN_n(R+R_i)\right]dR, \tag{43}$$

$$K_{23}^{mn}=\int_0^L a_3\varepsilon\alpha_1N_mN_n(R+R_i)dR, \tag{44}$$

$$K_{31}^{mn}=\int_0^L\left[-\frac{d^2}{dR^2}\{N_m(R+R_i)\}\frac{dN_n}{dR}+\frac{d}{dR}(2N_m)\frac{dN_n}{dR}+\left\{\frac{N_m}{(R+R_i)}\frac{dN_n}{dR}-\frac{N_mN_m}{(R+R_i)^2}\right\}\right]dR, \tag{45}$$

$$K_{32}^{mn}=\int_0^L\alpha_1\left[\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-N_m\frac{dN_n}{dR}\right]dR, \tag{46}$$

$$K_{33}^{mn}=\int_0^L\alpha_3\left[N_m\frac{dN_m}{dR}-\frac{d}{dR}\{N_m(R+R_i)\}\frac{dN_n}{dR}-\alpha_2 s(1+\tau^0 s)N_mN_n(R+R_i)\right]dR. \tag{47}$$

$$X=\begin{bmatrix}0\\0\\.\\.\\0\\0\end{bmatrix}, Y=\begin{bmatrix}0\\0\\.\\.\\0\\\frac{b\theta_b}{s}\end{bmatrix}, Z=\begin{bmatrix}0\\0\\.\\.\\0\\\frac{bP_b}{s}\end{bmatrix}. \tag{48}$$

V. PARTICULAR CASE (THERMOELASTIC MEDIUM)

In this section, we derive the results in case of absence of mass diffusion from our results as obtained above. So we neglect the diffusion effect by eliminating Eqs. (4) and (5) and by putting $c=0$ and $\beta_2=0$ into Eqs. (1), (3) and (8). The expression for displacement and temperature in thermoelastic medium can be written in the matrix form as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} \overline{U} \\ \overline{\theta} \end{bmatrix}=\begin{bmatrix} X \\ Y \end{bmatrix}, \tag{49}$$

where the sub-matrices $K_{11}, K_{12}, K_{21}, K_{22}, X, Y$ are given by Eqs. (39), (40), (42), (43) and (48) respectively.

When the medium is isotropic, taking $c_{11}=\lambda+2\mu, c_{12}=\lambda$ in Pal and Kanoria [25], the result agrees with Equation (49).

VI. NUMERICAL RESULTS AND DISCUSSIONS

For the sake of completeness we present some numerical applications to illustrate the analytical procedure as obtained earlier. As it is highly impossible to obtain the analytical solution of the problem in the physical domain (space-time domain), we develop a computer program by employing the numerical method of Laplace inversion given by Bellman et al.[29] and choose a time span given by seven values of time $t_i, i=1(1)7$ at which the field variables are evaluated from the negative of logarithms of the roots of the shifted Legendre polynomial of degree 7. We have made a comparative study for the variation of σ_R and P when $i=3,4,5,6,7,8$ for an isotropic medium in the context of GN-III model with $R=1.8$ which is shown in the table below:

Table 1: Comparison of radial stress and chemical potential for different values of i

i	Radial stress	Chemical potential
3	-0.07246	0.57625
4	-0.07625	0.60219
5	-0,07519	0.62491
6	-0.08241	0.66488
7	-0.08638	0.68912
8	-0.08638	0.68912

As seen from the table, for $i=7$ and $i=8$, the magnitude of σ_R and P are same. So, we take 7 integration points for sufficient accuracy of our computation. In the numerical computation, we have considered a copper-like material. The values of the material constants are taken to be:

$$K=386N/Ks, \alpha_t=1.78\times10^{-5}K^{-1}, \alpha_c=1.98\times10^{-4}m^3kg^{-1}, c_E=383.1m^2/K,$$
$$\mu=3.86\times10^{10}N/m^2, \lambda=7.76\times10^{10}N/m^2, \rho=8954kg/m^3, \tau^0=0.1\times10^{-4}s,$$
$$T_0=293K, \varepsilon=0.0168, \tau_T=0.15\times10^{-1}s, \tau_q=0.2\times10^{-1}s, \tau_\upsilon=0.1\times10^{-1}s,$$
$$D=0.9\times10^8kgm^{-3}s, c=1.2\times10^4m^2K^{-1}s^{-2}, d=0.9\times10^6m^5kg^{-1}s^{-2}$$

In G-N theory K^* is an additional material constant depending on the material. For copper material K^* is taken as $K^*=\frac{c_E(\lambda+2\mu)}{4}$. In order to study the effect of diffusion, we now present the thermophysical quantities in their graphical representations (figs. 1-6) under three-phase-lag model (3P model), Green-Naghdi III model (GN-III) and Lord-Shulman model (L-S) for a fixed time $t=0.14$. For the purpose of our analysis, we carry out our computation for the thermoelastic medium, that is, under the absence of diffusion (WOD) and as well as for the thermoelastic diffusive medium, that is, under the influence of diffusion (WD).

Figure-1 represents the variation of radial stress σ_R against the radial distance R under the effect of diffusion (WD) and without diffusion (WOD) for three-phase-lag model, GN-III model and L-S model. In each cases, the radial stress is noted to be zero at both the boundaries, which agrees with the boundary conditions. It is observed that, in presence of diffusion, the radial stress σ_R is reflexive throughout the medium for three-phase-lag model and in the region $1\le R\le1.4$ for GN-III model. In absence of diffusion, the radial stress σ_R is reflexive in the region $1\le R\le1.4$ for three-phase-lag model and in the region $1\le R\le1.2$ for GN-III model whereas σ_R is compressive in the rest portion of the region for both models. Under L-S model, the radial stress is fully compressive throughout the medium in all cases (WD and WOD).

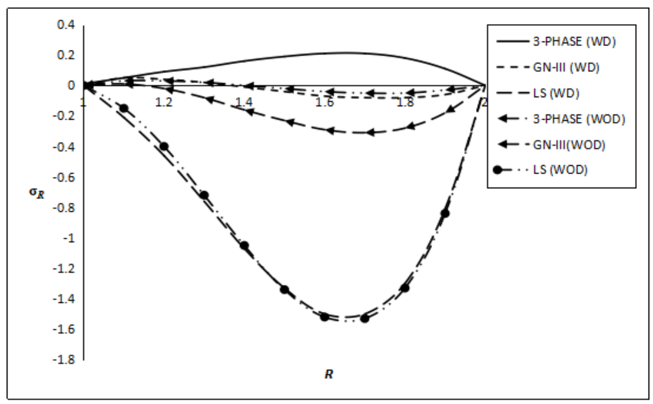

Fig. 1 Variation of radial stress

Figure-2 represents the variation of shear stress σ_θ against radial distance R in case of three different thermoelasticity models (3P model, GN-III, L-S model) in presence of diffusion (WD) and also in absence of diffusion (WOD). It is seen that the magnitude of σ_θ is maximum at the inner boundary $R=1$ for GN-III and L-S model. The effect of diffusion is very prominent in this medium for both Green-Naghdi III model (GN-III) and three-phase-lag model (3P model) whereas a mild effect of diffusion is observed under L-S model. Moreover the three-phase-lag theory predict a significantly higher numerical value as compared to other two models (GN-III, L-S).

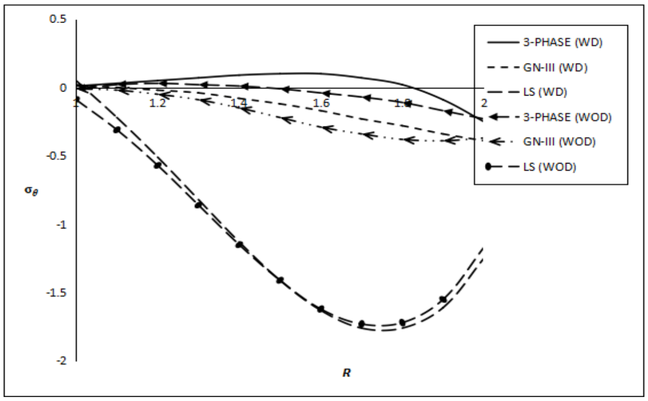

Fig. 2 Variation of shear stress

Figure-3 exhibits the space variation of the temperature (θ) considering for three different models (3P model, GN-III and L-S model). The effect of diffusion is not significant for this field under GN-III model though a mild effect of diffusion is observed under three-phase-lag model and L-S model. For GN-III model, the temperature fields increases with increase of radial distance throughout the medium. For three-phase-lag model, with the effect of diffusion, the temperature field initially remains at steady state, then after $R=1.4$ it increase rapidly with the increment of radial distance R. For L-S model, θ increases with the increase of radial distance and attains its maximum magnitude at the outer boundary for all the cases (WD and WOD). Moreover the three-phase-lag model predict a significantly higher numerical value as compared to other two models (GN-III, L-S).

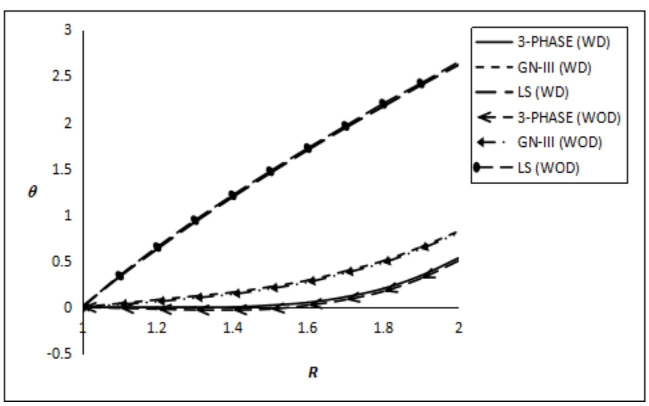

Fig. 3 Variation of temperature

Figure-4 is plotted to show the variation of displacement (U) versus radial distance R. The graph of displacement under three-phase-lag model and GN-III model for thermoelastic medium (WOD) are almost merged together, but with the effect of diffusion (thermoelastic diffusive medium) both the theory predict a significantly different value as compared to the previous case. For L-S model, the graph of displacement show significantly higher numerical value under thermoelastic diffusive medium (WD) as compared to the thermoelastic medium (WOD). For thermoelastic medium the displacement field shows its compressive nature in the range $1 \le R \le 1.76$ for GN-III model, $1 \le R \le 1.85$ for three-phase-lag model and $1 \le R \le 1.6$ for L-S model .

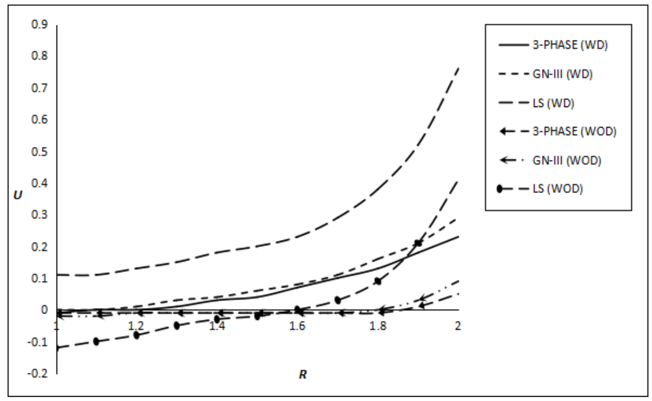

Fig. 4 Variation of displacement

Figure-5 shows the variation for chemical potential P for thermoelastic diffusive medium. The numerical results for the chemical potential are found to be in agreement with the theoretical boundary condition. The chemical potential shows a significantly different trend under L-S model as compared to GN-III and three-phase-lag models. For GN-III and three-phase-lag models, the chemical potential P is compressive in the region $1 \le R \le 1.4$ whereas for L-S model P is compressive in the region $1 \le R \le 1.45$.

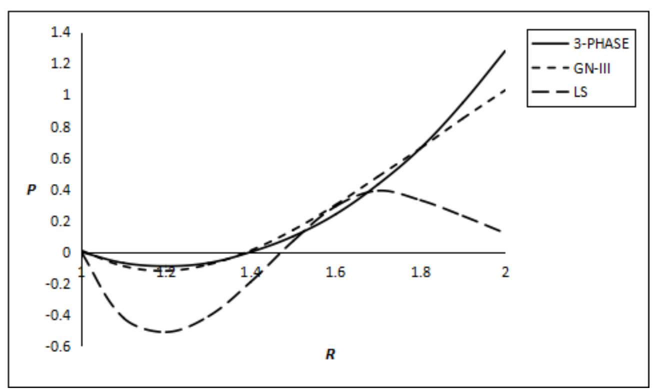

Fig. 5 Variation of chemical potential

Figure-6 shows the variation of mass concentration (C) for thermoelastic diffusive medium. Like the temperature field, under all three theories (3P model, GN-III and L-S model), the mass concentration shows minimum value at inner boundary and it increases with the increase of radial distance and attains its maximum value at the outer boundary of the cylinder. The mass concentration (C) is compressive throughout the considered region for all three models. It is also noted that the magnitude of the concentration is large for L-S model compared to GN-III model and 3P model.

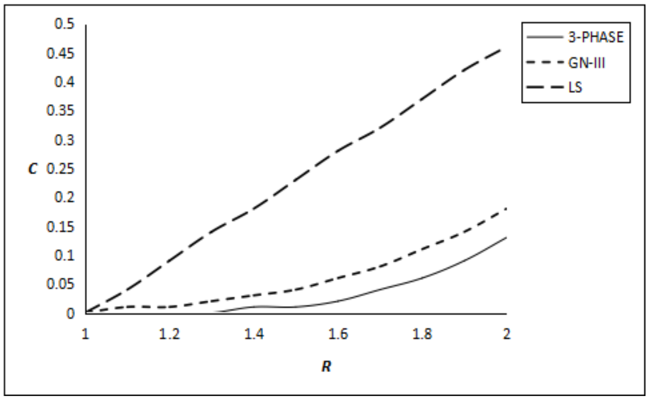

Fig. 6 Variation of concentration

VII. CONCLUSIONS

The results established in this paper can be summarized as follows:

(1) It is obvious that there is a distinct difference between the three-phase-lag model, GN-III model and L-S model for both thermoelastic medium (without diffusion) and thermoelastic diffusive medium.

(2) The effect of diffusion is significant on the solutions of radial stress, shear stress concentration and displacement field whereas it has mild effect on temperature and chemical potential field.

(3) It is believed that the result carried out through this problem should be beneficial for researchers working in the field of material science, the design of new materials, geophysics and other industrial applications.

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Authors Profile

Mr. Debarghya Bhattacharya received M.Sc degree in Applied Mathematics from University of Calcutta in 2011. Currently he is a Ph.D student in Department of Applied Mathematics, University of Calcutta, India. He has published 4 research papers in reputed international journals and it's also available online.

Dr. Prashanta Pal received his Ph.D degree in the field of applied mathematics from University of Calcutta in 2018. He has published more than 8 research papers in reputed international journals including Journal of thermal stresses, Acta Mechanica etc. and it's also available online.

Dr. Mridula Kanoria is currently a retired professor of Applied Mathematics at the Department of Applied Mathematics, University of Calcutta, 92 A.P.C. Road, Kol-700009, West Bengal, India. She received her Ph.D degree from University of Calcutta. She has more then 115 scientific publications in the field of solid mechanics.