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Solution of System of Fractional Differential Equations Using Variational Iteration Method

Shweta Pandey[1*], Sandeep Dixit[2]

[1]Dep.of Mathematics and Statistics, Gurukula Kangri Vishwavidyalaya,Haridwar-249404, INDIA
[2]Dep. of Mathematics, University of Petroleum and Energy Studies,Dehradun-248007, INDIA

[*]*Corresponding Author: shwetapandey154@gmail.com*

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***Abstract*—** In this paper, an approximate method to solve system of fractional differential equations analytically, fractional heat-like two-dimensional equation and stiff system of differential equations of nonlinear kind is obtained using variational iteration method (VIM). The results demonstrate that our approach is straightforward, effective and very simple. The numerical findings for different cases of problems are presented graphically. The outcomes reveal that the VIM is convenient, stable and performs extremely good in terms of simplicity and efficiency.

***Keywords*—** Variational iteration method; fractional heat-like two-dimensional equation; System of fractional differential equation; Caputo fractional derivative

I. INTRODUCTION

The study of arbitrary order integral and derivative which generalize integer-order differentiation and unifies n-fold integration is called fractional calculus. A reasonably sufficient study of fractional calculus can be found in [Samko et al. (1993)] and some of its applications in [Oldham and Spanier (1974)] and [Podlubny (1999)]. The problems directly using exact solutions of nonlinear partial differential equations are important for mathematical physics, these nonlinear problems are frequently associates with nonlinear wave equations which emerge in various fields for instance hydrodynamics, fluid mechanics, solid state physics, biology, optical fibers and plasma physics to further use them in the practical life and to better understand these phenomena it is necessary to look for their more accurate solutions. The solution of nonlinear equations conveniently and accurately can be obtained easily due to the flexibility and ability of the suggested method. The fractional differential equations models of problems in fluid flow, viscoelasticity, mathematical biology, the nonlinear oscillation of earthquake [He (1998)], engineering [Ferreira et al. (2008)], bioengineering [Magin (2004)] and electromechanical and electrical systems [Debnath (2003)] is used to successfully describe Partial fractional differential equations.

In general, there does not exist any analytical method to give an exact solution for fractional differential equations, using perturbation methods or linearization only approximate solutions can be calculated. Various methods are known till now to solve fractional and system of fractional differential equations, some of them are Homotopy perturbation method [Odibat and Momani(2008)], Adomian decomposition method [Jafari and Gejji(2006), Patel et.al.(2018)], Homotopy analysis method [Zurigat et al. (2010)]. The VIM was first proposed by [He (1998)] and has been effectively used in various problems like partial fractional differential equations of linear kind arising from fluid mechanics[Momani and Odibat (2006)], quadratic Riccati differential equation[Abbasbandy (2007)], parabolic partial differential equations [Javidi and Golbabai (2008)], parabolic integro-differential equations [Dehghan and Shakeri (2008)] occurring in heat conduction of materials, the generalized pantograph equation and wave like equations. The VIM has been turned out to be a strong mathematical tool for different kind of nonlinear and linear problems [Li Yun-dong and Yang Yi-ren (2017)] used VIM for conveying fluid pipe's vibration analysis, [Ghaneai and Hosseini (2016)] solved differential-algebraic equations through VIM, [Chen et al. (2017)] applied VIM to analyze free vibration problem of the rotating tapered Timoshenko beam. VIM needs no transformation or perturbation, discretization, linearization unlike the conventional numerical numerical methods. The implementations of the VIM methods are illustrated by [Ali and Malik (2014), Aski et. al. (2014), Chang (2016), Di Paola et. al., (2013), Siddiqi and Iftikhar (2015), Salkuyeh and Ali (2016)]. Some numerical illustrations are given to show the simplification and accuracy of the proposed algorithm. Some essential definitions and characteristics of fractional calculus theory are given which are used further for getting solutions.

The reliability and validity of this technique are tried by its relevance in different nonlinear fractional

differential equations, and the obtained solutions demonstrate that the proposed method is better in terms of accuracy and efficiency.

The paper is structured as follows, Section I contains the introduction of fractional calculus, Section II contain description of the VIM, Section III outlines the fundamental properties and definitions of fractional calculus, Section IV describes convergence and relevance of by numerical examples contain the numerical results, Section V presents the conclusion.

II. VARIATIONAL ITERATION METHOD

Variational Iteration method is the method of developing a correction functional using Lagrange multiplier, and the Lagrange multiplier is chosen so that its correction solution is better than the trial function or the initial approximation. We will take the subsequent general nonlinear system to demonstrate the fundamental concept of the VIM[Soltani and Shirzadi(2010)]

$$L[v(u)]+N[v(u)]=f(u), \tag{1}$$

where $f(u)$ is a given continuous function, N is a nonlinear operator and L is a linear operator. We may form a correction formula using VIM as follows:

$$v_{n+1}(u)=v_n(u)+\int_0^u \lambda(s)[Lv_n(s)+N\tilde{v}_n(s)-f(s)]ds, \tag{2}$$

where $\tilde{v}_n$ is treated as a restricted variation (the subscript 'n' denotes the n^{th} approximation) i.e. $\delta\tilde{v}_n=0$ and $v_0(u)$ is an initial approximation with possible unknowns and Lagrangian multiplier λ is calculated optimally via variational theory.

The consecutive iterations $v_n(u)$; $n\geq 1$, of the solution $v(x)$ will be readily accomplished with the help of calculated Lagrangian multiplier and by taking any selective function $v_0(u)$ therefore; the exact solution is acquired as

$$v(u)=\lim_{n\to\infty} v_n(u) \tag{3}$$

III. FRACTIONAL CALCULUS

In the last two centuries many studies and definitions of fractional calculus have been proposed which includes Weyl, Riemann-Liouville, Nashimoto and Caputo Compos, Reize fractional operators. A concise explanation of the fractional calculus definitions which are required for the further development is given below:

Definition 3.1. A real function $h(u), u>0$, is supposed to be in a space $C_\Omega, \Omega\in\Re$ for a real number $r(>\Omega)$ such that $h(u)=u^r h_1(u)$ where $h_1(u)\in C[0,\infty)$, and is supposed to be in C_Ω^b space. if $h^{(m)}\in C_\Omega, b\in N$.

Definition 3.2. The Riemann-Liouville fractional integral operator of order $\gamma\geq 0$, of a function $h\in C_\Omega$, $\Omega\geq -1$, is represented by

$$R^\alpha h(u)=\frac{1}{\Gamma(\gamma)}\int_0^u (u-t)^{\gamma-1}h(t)dt, \qquad \gamma>0, u>0,$$
$$R^0 h(u)=h(u) \tag{4}$$

Some of the characteristics of the operator R^γ are: [Podlubny (1999)]

For $h\in C_\Omega^b, \gamma,\nu>0, \Omega\geq -1$ and $\gamma\geq -1$,

1. $R^\gamma (u-a)^\nu=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\nu)}(u-a)^{\gamma+\nu}$
2. $R^\gamma R^\nu h(u)=R^{\gamma+\nu}h(u)$ □□□1□
3. $R^\nu R^\gamma h(u)=R^\gamma R^\nu h(u)$.

Definition 3.3. The fractional derivative of $h(u)$ in Caputo sense is characterized as:

$$D^\nu h(u)=R^{b-\nu}D^b h(u)=\frac{1}{\Gamma(m-\nu)}\int_0^u (u-t)^{b-\nu-1}h^b(t)dt \tag{5}$$

for $b-1<\nu\leq b, b\in N, u>0, h\in C_{-1}^b$.

The two basic properties of the Caputo's fractional derivative are also require here [Gorenflo and Mainardi(1997)].

Definition 3.4. If $b-1<\nu\leq b, b\in N$, $h\in C_{-1}^b$ and $\Omega\geq -1$, then $D^\nu R^\nu h(u)=h(u)$,

and

$$R^\nu D^\nu h(u)=h(u)-\sum_{k=0}^{b-1}h^{(k)}(0^+)\frac{(u-a)^k}{k!}, \ u>0. \tag{6}$$

The Caputo fractional derivative is used here since it allows conventional boundary and initial conditions to be included in the formulation of the problem.

Definition *3.5.* Caputo time fractional derivative operator of order $\gamma>0$; for the smallest integer 'b' that exceeds γ is given as follows

$$D_\gamma^t\xi(u,t)=\begin{cases}\frac{1}{\Gamma(b-\gamma)}\int_0^t (t-\tau)^{b-\gamma-1}\frac{\partial^b\xi(u,\tau)}{\partial t^b}, & \text{for } b-1<\gamma<b\\ \frac{\partial^b\xi(u,t)}{\partial t^b}, & \text{for } \gamma=b\in N.\end{cases} \tag{7}$$

We can consult [Podlubny (1999)] for mathematical properties of fractional integrals and derivatives.

IV. NUMERICAL RESULTS

We test both the convergence and accuracy of VIM utilized for the fractional systems of differential equations, by implementing it to the following problems. In this example, the exact solution of the system is not known, so to check the validity of the method, we define residual $E_b(r)$ as follows:

$$E_b(u) = (u_b(u) - u_{b-1}(u)) \tag{8}$$

Example 1: Let us consider the following fractional two-dimensional heat-like equation [(Golbabai and Javidi(2007))]

$$r_t^{\alpha} = r_{xx} + r_{yy} + A(x,y)r_x + B(x,y)r_y + r(x,y,t), \tag{9}$$

where $A(x,y) = k\sin kx\sin ky$ and $B(x,y) = k\cos kx\cos ky$ over the region $\Gamma = [0,1]\times[0,1]$ and $0 \le t \le 1$ with the following boundary conditions

$$r(0,y,t) = \sin t,\ r(1,y,t) = (1+y)\sin t,\ 0 < y < 1, t > 0$$
$$r(x,0,t) = \sin t,\ r(x,1,t) = (1+x)\sin t,\ \ 0 < x < 1, t > 0$$

with initial condition $r(x,y,0) = \sin kx\cos ky \quad 0 < x, y < 1$.

The exact solution $(\alpha = 1)$ was found to be

$$r(x,t) = e^{-2t}\sin x\cos y.$$

For this problem, making the above correction functional stationary, and noting that $\delta\tilde{u}_n = 0$ according to VIM, we obtain a correction functional as follows:

$$r_{n+1}(x,y,z,t) = r_n(x,y,z,t) - \int_0^t \lambda(s)\left(\frac{\partial^{\alpha}}{\partial s^{\alpha}} r_n(x,y,z,s) - \right.$$
$$\frac{\partial^2}{\partial x^2}\tilde{r}_n(x,y,z,s) - \frac{\partial^2}{\partial y^2}\tilde{r}_n(x,y,z,s) - A(x,y)\frac{\partial}{\partial x}\tilde{r}_n(x,y,z,s) \tag{10}$$
$$\left. -B(x,y)\frac{\partial}{\partial y}\tilde{r}_n(x,y,z,s) - \tilde{r}_n(x,y,z,s)\right) ds.$$

For $\alpha = 1$, we obtain the following stationary condition for Eq. (10)

$$1 + \lambda(t)\big|_{s=t} = 0, \qquad \lambda'(s) + 1 = 0 \tag{11}$$

The Lagrange multiplier $\lambda(s) = t - s - 1$ is obtained from Eq. (11) so we attain the iteration formula given as follows:

$$r_{n+1}(x,y,z,t) = r_n(x,y,z,t) - \int_0^t (t-s-1)\left(\frac{\partial^{\alpha}}{\partial s^{\alpha}} r_n(x,y,z,s) - \right.$$
$$\frac{\partial^2}{\partial x^2}\tilde{r}_n(x,y,z,s) - \frac{\partial^2}{\partial y^2}\tilde{r}_n(x,y,z,s) - A(x,y)\frac{\partial}{\partial x}\tilde{r}_n(x,y,z,s) \tag{12}$$
$$\left. -B(x,y)\frac{\partial}{\partial y}\tilde{r}_n(x,y,z,s) - \tilde{r}_n(x,y,z,s)\right) ds.$$

substituting the initial approximations $u_0(x,y,t) = u(x,y,0) = \sin k\,x\cos k\,y$, in Eq. (12), the various iterates are given as follows

$$r_0(x,y,t) = \sin kx\cos ky,$$
$$r_1(x,y,t) = (1-2t)\sin kx\cos ky,$$
$$r_2(x,y,t) = (1-k^2 - t + t^2 + 2k^2(t-1))\sin kx\cos ky,$$
$$r_3(x,y,t) = \left(1 - 6t + 6t^2 - \frac{4t^3}{3} - \frac{2t^{2-\alpha}}{\Gamma(4-2\alpha)} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right.$$
$$\left. + \frac{12t^{2-\alpha}}{\Gamma(4-\alpha)} - \frac{8t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{4t^{2-\alpha}\alpha}{\Gamma(4-\alpha)}\right)\sin kx\cos ky,$$

On taking $k = 1$, Fig. 1 shows the fourth order approximate solution $r_4(x,y,t)$ for $\alpha = 1$.

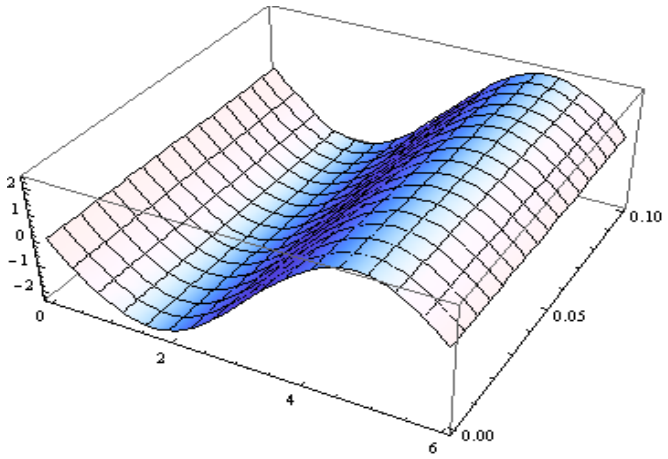

Figure 1. *The surface produced from $r_4(x,y,t)$ of variational iteration method for the parabolic partial differential equation with $\alpha = 1$.*

Example 2 The one-dimensional inhomogeneous fractional linear Burgers equation given by [Sakar et al. (2012)]

$$\frac{\partial^{\alpha} r}{\partial t^{\alpha}} + \frac{\partial r}{\partial x} - \frac{\partial^2 r}{\partial t^2} = \frac{2t^{2-\alpha}}{2\Gamma(3-\alpha)} + 2x - 2, \quad 0 < \alpha \le 1,\ t > 0,\ x \in R, \tag{13}$$

with initial condition

$$r(x,0) = x^2 \tag{14}$$

Applying VIM, the following iteration formula is attained

$$r_{n+1}(x) = r_n(x) - \int_0^x \left(\frac{\partial^{\alpha}}{\partial s^{\alpha}} r_n(s) + \frac{\partial^2}{\partial t^2}\tilde{r}_n(x,y,z,s)\right.$$
$$\left. + \frac{\partial}{\partial x}\tilde{r}_n(x,y,z,s) - \frac{2t^{2-\alpha}}{2\Gamma(3-\alpha)} - 2x + 2\right) ds \tag{15}$$

using the initial approximation, in (15), the various iterations are given as follows

$$r_1(x) = r_0(x) - \int_0^x \left(\frac{\partial^\alpha}{\partial s^\alpha} r_0(s) + \frac{\partial^2}{\partial t^2} \tilde{r}_0(x,y,z,s) + \frac{\partial}{\partial x} \tilde{r}_n(x,y,z,s) - \frac{2t^{2-\alpha}}{2\Gamma(3-\alpha)} - 2x + 2 \right) ds$$

$$r_1(x) = x^2 + t^2 + \frac{t^\alpha}{\Gamma(\alpha+1)}(2x-2) + 2xt - 2t,$$

$$r_2(x) = r_1(x) - \int_0^x \left(\frac{\partial^\alpha}{\partial s^\alpha} r_1(s) + \frac{\partial^2}{\partial t^2} \tilde{r}_1(x,y,z,s) + \frac{\partial}{\partial x} \tilde{r}_1(x,y,z,s) - \frac{2t^{2-\alpha}}{2\Gamma(3-\alpha)} - 2x + 2 \right) ds$$

$$r_2(x) = x^2 + t^2 + \frac{x^2}{8} - \frac{x^{3-\alpha}}{4\Gamma(3-\alpha)}, ...,$$

The approximate solutions $r_n(x)$ will converge, to the exact solutions $r(x)$ for $\alpha = 1$ as $n \to \infty$. $r(x) = \lim_{n\to\infty} r_n(x)$

$$r(x) = x^2 + t^2.$$

Example 3: In this example take the following system of nonlinear chemical reaction [Hell et al. (1972), Ganji et al.(2007)]

$$\begin{aligned} D_*^\alpha r &= -r, \\ D_*^\alpha s &= r - s^2, \qquad 0 < \alpha \le 1, \\ D_*^\alpha t &= t^2, \end{aligned} \tag{16}$$

where Caputo fractional derivative of order α is D^α, initial conditions $r(0) = 1,\ s(0) = 0,\ t(0) = 0$.

According to the VIM, the nonlinear expressions have to be measured as a restricted variation, so the correction functional is given as follows:

$$\begin{aligned} r_{n+1}(x) &= r_n(x) + \int_0^x \lambda_1(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} r_n(s) + \tilde{r}_n(s) \right) ds, \\ s_{n+1}(x) &= s_n(x) + \int_0^x \lambda_2(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} s_n(s) - \tilde{r}(s) + \tilde{s}_n^2(s) \right) ds, \\ t_{n+1}(x) &= t_n(x) + \int_0^x \lambda_3(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} t_n(s) - \tilde{s}_n^2(s) \right) ds. \end{aligned} \tag{17}$$

where $\lambda_1(s), \lambda_2(s)$ and $\lambda_3(s)$ are general Lagrange multipliers, and $\tilde{r}_n(s), \tilde{s}_n(s)$ denote the restricted variations, i.e. $\delta\tilde{r}_n(s) = \delta\tilde{s}_n^2 = 0$. Making the above correction functional stationary and using $\delta\tilde{r}_n(s) = \delta\tilde{s}_n^2 = 0$; we have

$$\begin{aligned} \delta r_{n+1}(x) &= \delta r_n(x) + \delta \int_0^x \lambda_1(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} r_n(s) \right) ds, \\ \delta s_{n+1}(x) &= \delta s_n(x) + \delta \int_0^x \lambda_2(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} s_n(s) \right) ds, \\ \delta t_{n+1}(x) &= \delta t_n(x) + \delta \int_0^x \lambda_3(s) \left(\frac{\partial^\alpha}{\partial s^\alpha} t_n(s) \right) ds. \end{aligned} \tag{18}$$

we acquire the subsequent stationary conditions on integrating (18) by parts and putting $\alpha = 1$

$$\begin{aligned} &\lambda_1'(s) = \lambda_2'(s) = \lambda_3'(s) = 0, \\ &1 + \lambda_1(s)\big|_{s=x} = 1 + \lambda_2(s)\big|_{s=x} = 1 + \lambda_3(s)\big|_{s=x} = 0. \end{aligned} \tag{19}$$

Solving Eq. (19), the Lagrange multipliers are obtained as

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = -1.$$

Thus, the subsequent iteration formulae is attained

$$\begin{aligned} r_{n+1}(x) &= r_n(x) - \int_0^x \left(\frac{\partial^\alpha}{\partial s^\alpha} r_n(s) + \tilde{r}_n(s) \right) ds, \\ s_{n+1}(x) &= s_n(x) - \int_0^x \left(\frac{\partial^\alpha}{\partial s^\alpha} s_n(s) - \tilde{r}(s) + \tilde{s}_n^2(s) \right) ds, \\ t_{n+1}(x) &= t_n(x) - \int_0^x \left(\frac{\partial^\alpha}{\partial s^\alpha} t_n(s) - \tilde{s}_n^2(s) \right) ds. \end{aligned} \tag{20}$$

substituting initial approximations $r_0(0) = 1,\ s_0(0) = 0,\ t_0(0) = 0$ in Eq. (20) the various iterates are given as follows

$$\begin{aligned} &r_1(x) = 1 - x, \quad r_2(x) = 1 - 2x + \frac{x^2}{2} + \frac{x^{2-\alpha}}{\Gamma(3-\alpha)}, ..., \\ &s_1(x) = x, \quad s_2(x) = 2x - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^{2-\alpha}}{\Gamma(3-\alpha)}, ..., \\ &t_1(x) = 0, \quad t_2(x) = \frac{x^3}{2}, ... \end{aligned} \tag{21}$$

The iteratives $r_n(x), s_n(x)$ and $t_n(x)$ represent the approximate solutions. Fig.2 shows the three respective residuals E_8 of the eighth order for $\alpha = 1.0$ whereas the Fig.3 depicts the three residuals E_4 of fourth order for $\alpha = 1.0$ and $\alpha = 0.99$.

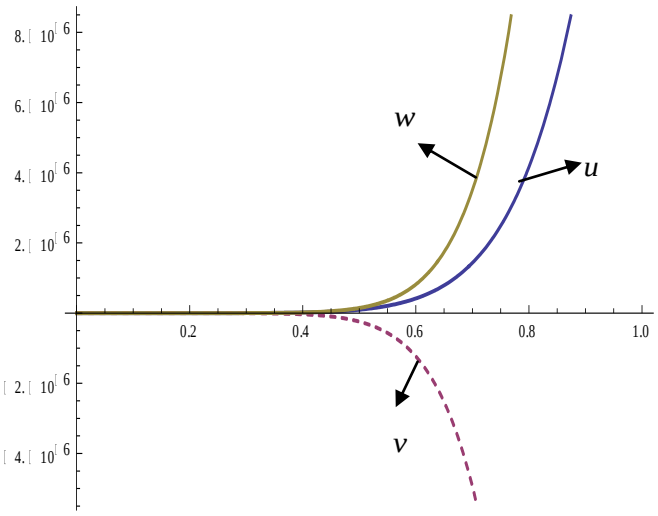

Figure 2. *Residuals for the eighth order approximate* $E_8, \alpha = 1.0$.

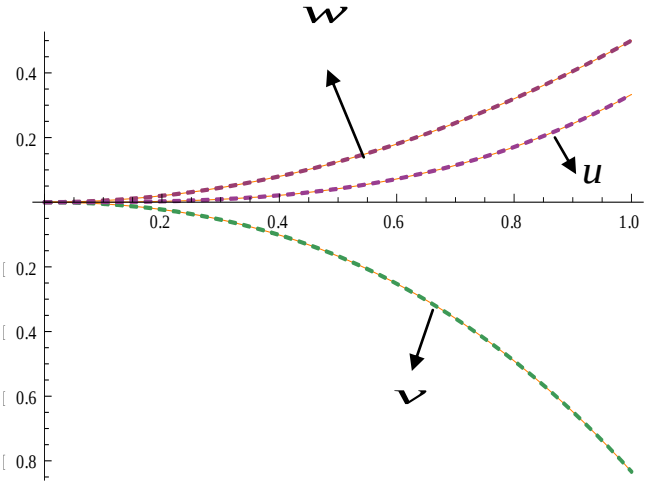

Figure 3. *Residuals* E_4 *smooth line:* $\alpha = 1.0$ *, dashed line:* $\alpha = 0.99$.

Example 4 Consider the following fractional stiff system of nonlinear differential equation:

$$\begin{aligned} D_*^{\alpha} u &= -102u + 100v^2, \\ D_*^{\alpha} v &= u - v - v^2, \end{aligned} \tag{22}$$

subject to the initial condition;

$$u(0) = 1 \qquad v(0) = 1 \tag{23}$$

For $\alpha = 1.0$; the accurate solution of this arrangement is $u(x) = e^{-2x}, v(x) = e^{-x}$. Using VIM, we derive the correction functional as

$$\begin{aligned} u_{n+1}(x) &= u_n(x) + \int_0^x \lambda_1(s)[\frac{\partial^{\alpha}}{\partial s^{\alpha}} u_n(s) + 102\tilde{u}_n(s) - 100\tilde{v}_n^2(s)]ds, \\ v_{n+1}(x) &= v_n(x) + \int_0^x \lambda_2(s)[\frac{\partial^{\alpha}}{\partial s^{\alpha}} v_n(s) - \tilde{u}_n(s) + \tilde{v}_n(s) + \tilde{v}_n^2(s)]ds. \end{aligned} \tag{24}$$

where restricted variations are $\tilde{u}_n(s), \tilde{v}_n(s)$ and $\tilde{v}_n^2(s)$, and $\lambda_1(s)$ and $\lambda_2(s)$ are Lagrangian multipliers, i.e. $\delta\tilde{u}_n(s) = \delta\tilde{v}_n(s) = \delta\tilde{v}_n^2(s) = 0.$

Again the general Lagrange multipliers $\lambda_1(s)$ and $\lambda_2(s)$ are found to be -1, thus giving the various approximations as

$$u_1(x) = 1 - 2x, \quad u_2(x) = 1 - 4x + 2x^2 + \frac{100x^3}{3} + \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}, ...,$$

$$v_1(x) = 1 - x, \quad v_2(x) = 1 - 2x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^{2-\alpha}}{\Gamma(3-\alpha)}, ... \ .$$

For $\alpha = 1,$ the solution is given by

$$u(x) = \lim_{n\to\infty} \sum_{m=0}^{n} u_m = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{(-2x)^m}{m!} = e^{-2x},$$

$$v(x) = \lim_{n\to\infty} \sum_{m=0}^{n} v_m = \lim_{n\to\infty} \sum_{m=0}^{n} \frac{(-x)^m}{m!} = e^{-x},$$

As $n \to \infty$, the approximate solutions u_n and v_n tend to exact solutions $u(x)$ and $v(x)$ respectively.

V. CONCLUSIONS

In this paper, the variational iteration method is utilized to get answers for systems of fractional differential equations of physical importance. The given numerical illustrations show that the variational iteration method is a very efficient and powerful method for finding exact and approximate solutions of high exactness for large number of problems and discretization of variables is not necessary. The numbers of iterations required to get the satisfactory result are very less, fractional approximate solutions converge to the exact solutions as fractional order derivative γ tends to their respective integral values thus demonstrating that the fractional order approximate solutions give a genuinely good idea of the probable exact solutions of the fractional systems.

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Authors Profile

Miss Shweta Pandey pursed Master of Science from Hemwati Nandan Bahuguna Garhwal University, in year 2015. She is currently pursuing Ph.D. under the supervision of Dr. S. R. Verma in Department of Mathematics and Statistics, Gurukula Kangri Vishwavidyalaya,Haridwar-249404, India and cosupervision of Dr. Sandeep Dixit Department of Mathematics, University of Petroleum and Energy Studies, Dehradun-248007, India.

Dr. Sandeep Dixit pursed Master of Science from Banars Hindu University in year 2005. He is completed his Ph.D. from IIT (BHU), Varanasi and currently working as Assistant Professor (SG) in Department of Mathematics, UPES, Dehradun since 2012. He has published more than 10 research papers in reputed international journals and conferences. His main research work focuses on wavelet based operational matrices, Singular integral equations, Iteration methods, and non linear differential equations. He has 9 years of teaching experience and 6 years of Research Experience.

