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Fixed-Point Theorems for *R*-Weakly Commuting Mappings on Parametric S-Metric Spaces

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***Abstract*—** In this paper, we prove some common fixed point theorems for variants of *R*-weakly commuting mappings in parametric *S*-metric spaces. Our proved results extend and generalized my known results in the area of fixed point theory. At the end of the paper, we give example to prove the validity of proved results. Our proved results have many applications in area of Non linear programming, fuzziness and intuitionistic fuzziness.

***Keywords*—** Parametric S-metric space, variants of R-weakly commuting mappings, fixed point.

I. INTRODUCTION

In 1922 Stefan Banach proved a common fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as Banach fixed point theorem or contraction mapping principle. Contractive conditions have been started by studying Banach's contraction principle. These contractive conditions have been used in various fixed-point theorems for some generalized metric spaces.
Recently, the notion of an S-metric has been studied by some mathematicians. This notion was introduced by Sedghi et al. in 2012 [1] as follows,

Definition 1.1 [1] Let X be a non-empty set and let *S*: $X \times X \times X \to [0,\infty)$ be a function. Then *S* is called an *S*-metric on X if,
(S1) $S(a,b,c)=0$ if an only if $a=b=c$,
(S2) $S(a,b,c) \le S(a,a,x)+S(b,b,x)+S(c,c,x)$,
for each $a,b,c,x \in X$.
The pair of (X,S) is called an *S*-metric space.

Definition 1.2 [2] Let X be a non-empty set and let $P: X \times X \times (0,\infty) \to [0,\infty)$ be a function. P is called a parametric metric on X if,
(P1) $P(a,b,t)=0$ if and only if $a=b$,
(P2) $P(a,b,t)=P(b,a,t)$,
(P3) $P(a,b,t) \le P(a,x,t)+P(x,b,t)$,
for each $a,b,x \in X$ and all $t > 0$.
The pair of (X,P) is called a parametric metric space.

Definition 1.3 [2,3] Let (X,P) be a parametric metric space andlet $\{a_n\}$ be a sequence in X :
(i) $\{a_n\}$ converges to x if and only if there exist $n_0 \in \mathbb{N}$ such that $P(a_n,x,t) < \varepsilon$, for all $n \ge n_0$ and all $t > 0$; that is, $\lim_{n\to\infty} P(a_n,x,t)=0$.
(ii) $\{a_n\}$ is called a Cauchy sequence if, for all $t > 0$, $\lim_{n,m\to\infty} P(a_n,a_m,t)=0$.
(iii) (X,P) is called complete if every Cauchy sequence is convergent.

Definition 1.4 [4] Let X be a non- empty set and let $P_S : X \times X \times X \times (0,\infty) \to [0,\infty)$ be a function. P_S is called a parametric *S*-metric on X if,
(PS1) $P_S(a,b,c,t)=0$ if and only if $a=b=c$,
(PS2)
$P_S(a,b,c,t) \le$
$P_S(a,a,x,t)+P_S(b,b,x,t)+P_S(c,c,x,t)$,
for each $a,b,c \in X$ and all $t > 0$.
The pair (X,P_S) is called a parametric *S*-metric space.

Lemma 1.1[4] Let (X, P_S) be a parametric S-metric space.then we have $P_S(a,a,b,t) = P_S(b,b,a,t),$ for each $a,b \in X$ and all $t > 0$.

Proof: using (PS2), we obtain,

$$P_S(a,a,b,t) \le 2P_S(a,a,b,t) + P_S(b,b,a,t)$$

$$= P_S(b,b,a,t),$$

$$P_S(b,b,a,t) \le 2P_S(b,b,b,t) + P_S(a,a,b,t)$$

$$= P_S(a,a,b,t).$$

From the above inequalities, we have

$$P_S(a,a,b,t) = P_S(b,b,a,t).$$

Definition 1.5[4] Let (X, P_S) be a parametric *S*-metric space and let $\{a_n\}$ be a sequence in *X*:

(i) $\{a_n\}$ converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $P_S(a_n, a_n, x, t) < \varepsilon$, for all $t > 0$; that is $\lim_{n\to\infty} P_S(a_n, a_n, x, t) = 0.$ It is denoted by $\lim_{n\to\infty} a_n = x.$

(ii) $\{a_n\}$ is called a Cauchy sequence if, for all $t > 0$,

$$\lim_{n,m\to\infty} P_S(a_n, a_n, a_m, t) = 0$$

(iii) (X, P_S) is called complete if every Cauchy sequence is convergent.

Lemma 1.2[2] Let (X, P_S) be a parametric S-metric space. If $\{a_n\}$ converges to x, then x is unique.

Proof: Let $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} a_n = y$ with $x \ne y.$ Then there exists $n_1, n_2 \in \mathbb{N}$ such that

$$P_S(a_n, a_n, x, t) < \frac{\varepsilon}{2},$$

$$P_S(a_n, a_n, y, t) < \frac{\varepsilon}{2},$$

For each ε > 0, all $t > 0$, and $n > n_1, n_2.$ if we take $n_0 = \max\{n_1, n_2\}$ then using (PS2) and Lemma 1.1 , we get

$$P_S(x,x,y,t) \le 2P_S(x,x,a_n,t) + P_S(y,y,a_n,t)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

For each $n \ge n_0$. Therefore $P_S(x,x,y,t) = 0$ and $x = y.$

II. MAIN RESULTS

In this section, we prove some fixed point results for *R*-weakly commuting mappings in parametric *S*-metric space.

Definition 2.1[5] A pair (f, g) of self-mappings of a parametric *S*-metric space (X, P_S) is said to be *R*-weakly commuting at a point $a \in X$ if

$$P_S(fga, fga, gfa, t) \le R\ P_S(fa, fa, ga, t)$$

for some $R > 0$.

Definition 2.2[6,7] A pair (f, g) of self-mappings of a parametric *S*-metric space (X, P_S) is said to be pointwise *R*-weakly commuting on *X* if given $a \in X$, there exists $R > 0$ such that

$$P_S(fga, fga, gfa, t) \le R\ P_S(fa, fa, ga, t).$$

Theorem 2.1 Let (X, P_S) be a complete parametric *S*-metric space and let f and g be *R*-weakly commuting self-mapping of X satisfying the following conditions:

(i) $f(X) \subseteq g(X)$;

(ii) f or g is continuous;

(iii) $P_S(fa, fa, fb, t) \le q\ P_S(ga, ga, gb, t)$,

for every $a,b \in X$ and $0 \le q < 1$.

Then f and g have a unique fixed point in X.

Proof: Let a_0 be an arbitrary point in *X*. By (i) one can choose a point $a_1 \in X$ such that $fa_0 = ga_1$. In general, choose a_{n+1} such that $b_n = fa_n = ga_{n+1}$.

Now we show that $\{b_n\}$ is a P_S - Cauchy sequence in *X*. For proving this we take $a = a_n, b = a_{n+1}$ in (iii), we have

$$P_S(fa_n, fa_n, fa_{n+1}, t) \le q.P_S(ga_n, ga_n, ga_{n+1}, t)$$

$$= q\ P_S(fa_{n-1}, fa_{n-1}, fa_n, t)$$

Continuing in the same way, we get

$$P_S(fa_n, fa_n, fa_{n+1}, t) \le q^n.P_S(fa_0, fa_0, fa_1, t).$$

This implies,

$$P_S(b_n, b_n, b_{n+1}, t) \le q^n P_S(b_0, b_0, b_1, t).$$

Therefore, for all $n, m \in N,\ n < m$,

$P_S(b_n,b_n,b_m,t) \le P_S(b_n,b_n,b_{n+1},t)$ + $P_S(b_{n+1},b_{n+1},b_{n+2},t)$ +……+ $P_S(b_{m-1},b_{m-1},b_m,t)$

$\le (q^n + q^{n+1} + ... + q^{m-1})\ P_S(b_0,b_0,b_1,t)$

$\le (q^n + q^{n+1} + ...)\ P_S(b_0,b_0,b_1,t)$

$= \dfrac{q^n}{(1-q)}\ P_S(b_0,b_0,b_1,t) \to 0$ as $n \to \infty$.

Thus $\{b_n\}$ is a P_S- Cauchy sequence in *X*. since (X,P_S) is complete S-metric space, therefore, there exists a point $c \in X$ such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} ga_n = \lim_{n\to\infty} fa_n = c$. Let us suppose that the mapping f is continuous. Therefore $\lim_{n\to\infty} fga_n = \lim_{n\to\infty} ffa_n = fc$.

Since f and g are *R*-weakly commuting,

$P_S(fga_n, fga_n, gfa_n, t) \le R\ P_S(fa_n, fa_n, ga_n, t)$, where R > 0.

On letting $n \to \infty$ we get $\lim_{n\to\infty} gfa_n = \lim_{n\to\infty} fga_n = fc$.

Now we prove that $c = fc$. Suppose $c \ne fc$ then $P_S(c, fc, fc, t) > 0$

On setting $a = a_n, b = fa_n$ in (iii), we have

$$P_S(fa_n, fa_n, ffa_n, t) \le q\ P_S(ga_n, ga_n, gfa_n, t)$$

Letting limit as $n \to \infty$ we get

$P_S(c,c,fc,t) \le q\ P_S(c,c,fc,t) < P_S(c,c,fc,t)$.

Is a contradiction.

Therefore $c = fc$ since $f(X) \subseteq g(X)$ we can find $c_1 \in X$ such that $c = fc = gc_1$.

Now put $a = fa_n, b = c = c_1$ in (iii), we have

$P_S(ffa_n, ffa_n, fc_1, t) \le q\ P_S(gfa_n, gfa_n, gc_1, t)$.

Taking limit as $n \to \infty$ we get

$$P_S(fc, fc, fc_1, t) \le q.P_S(fc, fc, gc_1, t)$$
$$= q\ P_S(fc, fc, fc, t) = 0,$$

Which implies that $fc = fc_1$ i.e., $c = fc = fc_1 = gc_1$.

Also by using the definition og *R*-weakly commutativity,

$$P_S(fc, fc, gc, t) = P_S(fgc_1, fgc_1, gfc_1, t)$$
$$\le R.P_S(fc_1, fc_1, gc_1, t) = 0,$$

Implies that $fc = gc = c$. Thus c is a common fixed point of f and g.

Uniqueness: assume that $d(\ne c)$ be another common fixed point of f and g.

Then $P_S(c,c,d,t) > 0$ and

$$P_S(c,c,d,t) = P_S(fc, fc, fd, t) \le q\ P_S(gc, gc, gd, t).$$
$$= q\ P_S(c,c,d,t) < P_S(c,c,d,t).$$

Which is a contradiction. Therefore $c = d$. Hence uniqueness follows.

Example 2.1 Let $X = R$ and let the function $P_S : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$P_S(a,b,c,t) = g(t)(|a-b| + |b-c| + |a-c|$

for each $a,b,c \in R$ and all t > 0, where $g:(0,\infty) \to (0,\infty)$ is a continuous function. Then P_S is a parametric *S*-metric and the pair (R,P_S) is a parametric *S*-metric space. Define self-mappings *f* and *g* on *X* by $fx = x$ and $gx = 2x-1$, for all $x \in X$. Here we note that,

(i) $f(X) \subseteq g(X)$;

(ii) *f* is continuous on *X*;

(iii) $P_S(fa, fb, fc, t) \le q\ P_S(ga, gb, gc, t)$, holds for all $a,b,c \in X$, $\frac{1}{2} \le q < 1$.

Further, the mappings *f* and *g* are *R*-weakly commuting. Thus, all conditions of the theorem 2.1 are satisfied and $x = 1$ is the unique common fixed point of *f* and *g*.

III. Conclusion and Future Scope

In this paper, we prove some common fixed point theorems for variants of *R*-weakly commuting mappings in parametric *S*-metric spaces. Our proved results extend and generalized my known results in the area of fixed point theory. At the end of the paper, we give example to prove the validity of proved results. In future, above proved results applied to various other metric spaces such as fuzzy metric[8], modified fuzzy metric[9,10] and complex metric space[11]. Our proved results have many applications in area of Non linear programming, fuzziness and intuitionistic fuzziness [12-14].

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