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Commutative Monoid of Pythagorean Fuzzy Matrices

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Abstract— In this paper, we prove the set of all Pythagorean fuzzy matrices form a commutative monoid with respect to algebraic sum and algebraic product. Also, the De Morgan's laws and Distributive laws are provided and we define the @ operations on Pythagorean fuzzy matrices and analyze its algebraic properties. Further, some results prove equalities and inequalities of Pythagorean fuzzy matrices.

Keywords—Intuitionistic fuzzy matrix, Pythagorean fuzzy set, Pythagorean fuzzy matrix, Algebraic sum and Algebraic product.

I. Introduction

The concept of intuitionistic fuzzy matrix (IFM) was introduced by Pal [4] and simultaneously by Im et.al [2] to generalize the concept of Thomason's [19] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\left\langle a_{ij}, a'_{ij} \right\rangle$. The sum $a_{ij} + a'_{ij}$ of each ordered pair is less than or equal to 1. Since the appearance of IFM in 2001, several researchers [9,10,18] have importantly contributed to the development of IFM theory and its application, resulting in greater success from the theoretical and technological points of view. Muthuraji et.al [8] obtain a decomposition of intuitionistic fuzzy matrix. In [2,3] the concept of the determination theory and the adjoint of a square IFM were studied. Also, they investigated their properties.Pal [11] introduced the Intuitionistic fuzzy determinant and [5] defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product etc. and proved equality between IFMs. Mondal and Pal [6] studied the similarity relations , together with invertibility conditions and eigenvalues of IFMs. Emam and Fndh [1] defined some kinds of IFMs, the max-min and min-max composition of IFMs. Also, they derived several important results of these compositions and construct an idempotent IFM from any given one through the min-max composition. Zhang [24] studied intuitionistic fuzzy value and introduced the concept of composition two intuitionistic fuzzy matrices. Sriram and Boobalan [18] studied the properties of algebraic sum and algebraic product of intuitionistic fuzzy matrices and prove that the set of all intuitionistic fuzzy matrices form a commutative monoid. In[14], we defined Hamacher operations on fuzzy matrices and investigated their algebraic properties, they [15] extended Hamacher operations to IFMs. Yager [20,21,22] introduced the concept of the Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. The PFS characterized by a membership degree and a nonmembership degree satisfying the condition that the square sum of its membership degree and nonmembership degree is equal to or less than 1, has much stronger ability than IFS to model such uncertain information in MCDM problems. Zhang and Xu [25] defined some novel operational laws of PFS and discuss its desirable properties. In [16], we defined the Pythagorean fuzzy matrices and their basic operations, they construct nA and A^n of a PFM A and them desirable properties.

II. PRELIMINARIES

In this section, the basic concept of Intuitionistic fuzzy matrix (IFM) , Pythagorean fuzzy set (PFS) and Pythagorean fuzzy matrices (PFMs).

Definition 2.1. [4] An intuitionistic fuzzy matrix (IFM) is a matrix of pairs $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right)$ of a nonnegative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satisfying $0 \le a_{ij} + a'_{ij} \le 1$ for all i, j.

Definition 2.2.[25]Let a set X be a universe of discourse A Pythagorean fuzzy set (PFS) P is an object having the form $P = \left(\left\langle x, P(\mu_p(x), \nu_p(x)) \mid (x \in X) \right\rangle\right)$, where the function $\mu_p : X \to [0,1]$ defines the degree of membership and $\nu_p : X \to [0,1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively, and for every $x \in X$, it holds that $(\mu_p(x))^2 + (\nu_p(x))^2 \leq 1$.

Definition. 2.3. [17] Let $A = (\langle a_{ij}, a'_{ij} \rangle)$ and $B = (\langle b_{ij}, b'_{ij} \rangle)$ be two intuitionistic fuzzy matrices of same size $m \times n$. Then

$(i) A \vee B = (\langle \max\{a_{ij}, b_{ij}\} \min\{a'_{ij}, b'_{ij}\} \rangle)$

$(ii) A \wedge B = (\langle \min\{a_{ij}, b_{ij}\} \max\{a'_{ij}, b'_{ij}\} \rangle)$

$(iii) A^C = (\langle a'_{ij}, a_{ij} \rangle)$ (the complement of A)

$(iv) A \leq B$ if and only if $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.

$(v) A \oplus B = (\langle a_{ij} + b_{ij} - a_{ij}b_{ij}, a'_{ij}b'_{ij} \rangle)$ is called the algebraic sum of A and B.

$(vi) A \odot B = (\langle a_{ij}b_{ij}, a'_{ij} + b'_{ij} - a'_{ij}b'_{ij} \rangle)$ is called the algebraic product of A and B.

Definition 2.4. [16] A Pythagorean fuzzy matrix (PFM) is a pair $A = (\langle a_{ij}, a'_{ij} \rangle)$ of non negative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satisfying $a_{ij}^{\ 2} + a'^{\,2}_{ij} \leq 1$, for all i, j.

Analogously Definition 2.3, we defined the following operations on PFMs [16].

Definition 2.5.Given three PFMs, A, B and C of the basic operations are

$(i) A \vee B = (\langle \max\{a_{ij}, b_{ij}\}, \min\{a'_{ij}, b'_{ij}\} \rangle)$

$(ii) A \wedge B = (\langle \min\{a_{ij}, b_{ij}\}, \max\{a'_{ij}, b'_{ij}\} \rangle)$

$(iii) A^C = (\langle a'_{ij}, a_{ij} \rangle)$ (the complement of A)

$(iv) A \leq B$ if and only if $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.

$(v) A \oplus_P B = \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, a'_{ij}b'_{ij} \right\rangle\right)$

$(vi) A \odot_P B = \left(\left\langle a_{ij}b_{ij}, \sqrt{a'^{\,2}_{ij} + b'^{\,2}_{ij} - a'^{\,2}_{ij} b'^{\,2}_{ij}} \right\rangle\right)$, where $+, -$ and $.$ are ordinary addition, subtraction and multiplication respectively.

Definition 2.6.The $m \times n$ zero PFM O is a PFM all of whose entries are $\langle 0,1 \rangle$. The $m \times n$ universal PFM J is a PFM all of whose entries are $\langle 1,0 \rangle$.

III. SOME PROPERTIES OF PYTHAGOREAN FUZZY MATRICES

In this section, new equalities and inequalities are obtained and proved by means of some Pythagorean fuzzy operations ($\oplus_P$,$\odot_P$, $\vee$, $\wedge$).

The relation between $\oplus_P$ and $\odot_P$ is established by the following theorem.

Theorem 3.1.If A and B are two PFMs, then $A \oplus_P B \geq A \odot_P B$.

Proof. The ij^{th} element of $A \oplus_P B = \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, a'_{ij}b'_{ij} \right\rangle\right)$,

$$A \odot_P B = \left(\left\langle a_{ij}b_{ij}, \sqrt{a'^{\,2}_{ij} + b'^{\,2}_{ij} - a'^{\,2}_{ij} b'^{\,2}_{ij}} \right\rangle\right)$$

Assume that $\sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} \geq a_{ij}b_{ij}$.

$\Rightarrow a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2 \geq a_{ij}^2 b_{ij}^2$

$\Rightarrow a_{ij}^2(1 - b_{ij}^2) + b_{ij}^2(1 - a_{ij}^2) \geq 0$ Which is true as $a_{ij}, b_{ij} \in [0,1]$

Similarly, we can prove $a'_{ij}b'_{ij} \leq \sqrt{a'^{\,2}_{ij} + b'^{\,2}_{ij} - a'^{\,2}_{ij} b'^{\,2}_{ij}}$

Hence, $A \oplus_P B \geq A \odot_P B$

Theorem 3.2.If A is any PFM,then

$(i)(A \oplus_P A) \geq A$

$(ii)(A \odot_P A) \leq A$

Proof. (i) The ij^{th} element of $A \oplus_P A$ is $\left\langle \sqrt{a_{ij}^2 + a_{ij}^2(1 - a_{ij}^2)}, a'^{\,2}_{ij} \right\rangle$

Since $\sqrt{a_{ij}^2 + a_{ij}^2(1 - a_{ij}^2)} \geq \sqrt{a_{ij}^2} \geq a_{ij}$

Also, $a'^{\,2}_{ij} \leq a'_{ij}$

$\Rightarrow (A \oplus_P A) \geq A$

(ii) It can be proved analogously.

The operators $\oplus_P$ and $\odot_P$ are commutative as well as associative.

Theorem 3.3.If A and B are two PFMs, then

$(i) A \oplus_P B = B \oplus_P A$

$(ii) A \odot_P B = B \odot_P A$

$(iii) (A \oplus_P B) \oplus_P C = A \oplus_P (B \oplus_P C)$

$(iv) (A \odot_P B) \odot_P C = A \odot_P (B \odot_P C)$

Proof. The proof of (i) and (ii) are trivial.

$(iii) (A \oplus_P B) \oplus_P C$

$$= \left(\left\langle \sqrt{(a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2) + c_{ij}^2 - (a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2) c_{ij}^2}, \; a'_{ij} b'_{ij} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 - a_{ij}^2 b_{ij}^2 c_{ij}^2 - a_{ij}^2 b_{ij}^2 - b_{ij}^2 c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}, \; a'_{ij} b'_{ij} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 - a_{ij}^2 b_{ij}^2 c_{ij}^2 - a_{ij}^2 c_{ij}^2 - b_{ij}^2 c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}, \; a'_{ij} b'_{ij} c'_{ij} \right\rangle \right) \quad (3.1)$$

$A \oplus_P (B \oplus_P C)$

$$= \left(\left\langle \sqrt{a_{ij}^2 + (b_{ij}^2 + c_{ij}^2 - b_{ij}^2 c_{ij}^2) - a_{ij}^2 (b_{ij}^2 + c_{ij}^2 - b_{ij}^2 c_{ij}^2)}, \; a'_{ij} b'_{ij} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 + c_{ij}^2 - a_{ij}^2 b_{ij}^2 c_{ij}^2 - a_{ij}^2 c_{ij}^2 - b_{ij}^2 c_{ij}^2 + a_{ij}^2 b_{ij}^2 c_{ij}^2}, \; a'_{ij} b'_{ij} c'_{ij} \right\rangle \right) \quad (3.2)$$

From (3.1) and (3.2), we get result (iii).

(iv) It can be proved analogously.

The operators $\oplus$ and $\odot$ do not obey the De Morgan's laws over transpose.

Theorem 3.4.If A and B are two PFMs, then

$(i) (A \oplus_P B)^T = A^T \oplus_P B^T$

$(ii) (A \odot_P B)^T = A^T \odot_P B^T$

(iii) If $A \le B$,then $(A \oplus_P C) \le (B \oplus_P C)$ and $(A \odot_P C) \le (B \odot_P C)$

Proof. The proof of (i) and (ii) are trivial.

(iii) Since $A \le B, a_{ij} \le b_{ij}$ and $a'_{ij} \ge b'_{ij}$

Then $\sqrt{a_{ij}^2 (1 - a_{ij}^2)} \le \sqrt{b_{ij}^2 (1 - c_{ij}^2)}, a'_{ij} c'_{ij} \ge b'_{ij} c'_{ij}$

$\Rightarrow \sqrt{a_{ij}^2 + c_{ij}^2 - a_{ij}^2 c_{ij}^2} \le \sqrt{b_{ij}^2 + c_{ij}^2 - b_{ij}^2 c_{ij}^2}, a'_{ij} c'_{ij} \ge b'_{ij} c'_{ij}$ for all i, j.

Hence $(A \oplus_P C) \le (B \oplus_P C)$

Similarly, we prove $(A \odot_P C) \le (B \odot_P C)$

Theorem 3.5.If A and B are two PFMs, then

$(i) A \oplus_P B \ge (A \vee B)$

$(ii) A \odot_P B \le (A \vee B)$

Proof. (i) Let $\langle c_{ij}, c'_{ij} \rangle$ and $\langle d_{ij}, d'_{ij} \rangle$ be the ij^{th} elements of the PFMs $A \oplus_P B$ and $A \vee B$ respectively.

Now, $c_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} = \begin{cases} \sqrt{a_{ij}^2 + b_{ij}^2 (1 - a_{ij}^2)} \ge a_{ij} \\ \sqrt{b_{ij}^2 + a_{ij}^2 (1 - b_{ij}^2)} \ge b_{ij} \end{cases}$

$c_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} \ge \max\{a_{ij}, b_{ij}\} = d_{ij}$,

$c'_{ij} = a'_{ij} b'_{ij} \le \min\{a'_{ij}, b'_{ij}\} = d'_{ij}$

Thus, $c_{ij} \ge d_{ij}$ and $c'_{ij} \le d'_{ij}$ for all i, j.

Hence, $A \oplus_P B \ge (A \vee B)$.

(ii) It can be proved analogously.

Theorem 3.6.If A and B are two PFMs, then

$(i) A \oplus_P B \ge (A \wedge B)$

$(ii) A \odot_P B \le (A \wedge B)$

Proof. (i)Let $\langle c_{ij}, c'_{ij} \rangle$ and $\langle d_{ij}, d'_{ij} \rangle$ be the ij^{th} elements of the PFMs $A \oplus_P B$ and $A \wedge B$ respectively.

Now,

$$c_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} = \begin{cases} \sqrt{a_{ij}^2 + b_{ij}^2 (1 - a_{ij}^2)} \ge a_{ij} \\ \sqrt{b_{ij}^2 + a_{ij}^2 (1 - b_{ij}^2)} \ge b_{ij} \end{cases}$$

$c_{ij} = \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}$

$\ge \min\{a_{ij}, b_{ij}\} = d_{ij}, c'_{ij} = a'_{ij} b'_{ij} \le \max\{a'_{ij}, b'_{ij}\} = d'_{ij}$

Thus, $c_{ij} \ge d_{ij}$ and $c'_{ij} \le d'_{ij}$ for all i, j.

Hence, $A \oplus_P B \ge (A \wedge B)$

(ii)It can be proved analogously.

Remark 3.7.For $a, b \in [0,1]$.

Then, $ab \le \frac{a+b}{2}, \frac{a+b}{2} \le a + b - ab.$

Theorem 3.8.If A and B are two PFMs, then

$(i)(A \oplus_P B) \wedge (A \odot_P B) = A \odot_P B$

$(ii)(A \oplus_P B) \vee (A \odot_P B) = A \oplus_P B$

Proof.

$(i)(A \oplus_P B) \wedge (A \odot_P B)$

$$= \left(\left\langle \begin{array}{l} \min\left\{ \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, a_{ij} b_{ij} \right\}, \\ \max\left\{ a'_{ij} b'_{ij}, \sqrt{a'^2_{ij} + b'^2_{ij} - a'^2_{ij} b'^2_{ij}} \right\} \end{array} \right\rangle \right)$$

$$= \left(\left\langle a_{ij} b_{ij}, \sqrt{a'^2_{ij} + b'^2_{ij} - a'^2_{ij} b'^2_{ij}} \right\rangle \right)$$

$$= A \odot_P B$$

(ii) It can be proved analogously.

We shall next examine the absorption and distributive properties for PFMs under the operations $\oplus_P$ and $\odot_P$ which are combined with $\wedge$ and $\vee$.

Theorem 3.9.If A and B are two PFMs, then

$(i) A \oplus_P (A \odot_P B) \geq A$

$(ii) A \odot_P (A \oplus_P B) \geq A$

Proof.

$(i) A \oplus_P (A \odot_P B)$

$$= \left(\left\langle \begin{array}{l} \sqrt{a_{ij}^2 + a_{ij}^2 b_{ij}^2 - a_{ij}^2 (a_{ij}^2 b_{ij}^2)}, \\ a'_{ij} (\sqrt{a'^2_{ij} + b'^2_{ij} - a'^2_{ij} b'^2_{ij}}) \end{array} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{a_{ij}^2 + a_{ij}^2 b_{ij}^2 (1 - a_{ij}^2)}, a'_{ij} (\sqrt{1 - (1 - a'_{ij})(1 - b'_{ij})}) \right\rangle \right)$$

$$\geq A$$

(ii) It can be proved analogously.

We discuss the Distributivity law in the case the operation of algebraic sum, algebraic product, $\vee$ and $\wedge$ are combined with each other.

Theorem 3.10.If A , B and C are three PFMs, then

$(i)(A \vee B) \oplus_P C = (A \oplus_P C) \vee (B \oplus_P C)$

$(ii)(A \wedge B) \oplus_P C = (A \oplus_P C) \wedge (B \oplus_P C)$

Proof.

$(i)(A \vee B) \oplus_P C$

$$= \left(\left\langle \begin{array}{l} \sqrt{\max\left\{ a_{ij}^2, b_{ij}^2 \right\} + c_{ij}^2 - \max\left\{ a_{ij}^2, b_{ij}^2 \right\} c_{ij}^2}, \\ \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \end{array} \right\rangle \right) \quad (3.3)$$

$(A \oplus_P C) \vee (B \oplus_P C)$

$$= \left(\left\langle \begin{array}{l} \sqrt{\max\left\{ a_{ij}^2 + c_{ij}^2 - a_{ij}^2 c_{ij}^2, b_{ij}^2 + c_{ij}^2 - b_{ij}^2 c_{ij}^2 \right\}}, \\ \min\left\{ a'_{ij} c'_{ij}, b'_{ij} c'_{ij} \right\} \end{array} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\max\left\{ a_{ij}^2 (1 - c_{ij}^2) + c_{ij}^2, b_{ij}^2 (1 - c_{ij}^2) + c_{ij}^2 \right\}}, \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\max\left\{ a_{ij}^2 (1 - c_{ij}^2), b_{ij}^2 (1 - c_{ij}^2) \right\} + c_{ij}^2}, \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\max\left\{ a_{ij}^2, b_{ij}^2 \right\} (1 - c_{ij}^2) + c_{ij}^2}, \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\max\left\{ a_{ij}^2, b_{ij}^2 \right\} - \max\left\{ a_{ij}^2, b_{ij}^2 \right\} c_{ij}^2 + c_{ij}^2}, \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\max\left\{ a_{ij}^2, b_{ij}^2 \right\} + c_{ij}^2 - \max\left\{ a_{ij}^2, b_{ij}^2 \right\} c_{ij}^2}, \min\left\{ a'_{ij}, b'_{ij} \right\} c'_{ij} \right\rangle \right) (3.4)$$

From (3.3) and (3.4), we get the result (i).

(ii) It can be proved analogously.

Theorem 3.11.If A , B and C are three PFMs, then

$(i) A \oplus_P (B \vee C) = (A \oplus_P B) \vee (A \oplus_P C)$

$(ii) A \odot_P (B \vee C) = (A \odot_P B) \vee (A \odot_P C)$

Proof.

$(i) A \oplus_P (B \vee C)$

$$= \left(\left\langle \begin{array}{l} \sqrt{a_{ij}^2 + \max\left\{ b_{ij}^2, c_{ij}^2 \right\} - a_{ij}^2 \max\left\{ b_{ij}^2, c_{ij}^2 \right\}}, \\ a'_{ij} \max\left\{ b'_{ij}, c'_{ij} \right\} \end{array} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, \min\left\{ a'_{ij} b'_{ij}, a'_{ij} c'_{ij} \right\} \right\rangle \right)$$

If $b_{ij} > c_{ij}$, then

$A \oplus_P (B \vee C)$

$= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, \min\left\{ a'_{ij} b'_{ij}, a'_{ij} c'_{ij} \right\} \right\rangle \right)$

$(A \oplus_P B) \vee (A \oplus_P C)$

Also $= \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, \min\left\{ a'_{ij} b'_{ij}, a'_{ij} c'_{ij} \right\} \right\rangle \right)$

Since $b_{ij} > c_{ij}$, then $b_{ij}(1 - a_{ij}) > c_{ij}(1 - a_{ij})$

Thus, $\sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} > \sqrt{a_{ij}^2 + c_{ij}^2 - a_{ij}^2 c_{ij}^2}$

Similarly, if $b_{ij} \le c_{ij}$,

then $A \oplus_P (B \vee C) = (A \oplus_P B) \vee (A \oplus_P C)$

Hence, $A \oplus_P (B \vee C) = (A \oplus_P B) \vee (A \oplus_P C)$

(ii) It can be proved analogously.

The following results are obvious.

Theorem 3.12.If A , B and C are three PFMs, then

$(i) A \oplus_P (B \wedge C) = (A \oplus_P B) \wedge (A \oplus_P C)$

$(ii) A \odot_P (B \wedge C) = (A \odot_P B) \wedge (A \odot_P C)$

$(iii)(A \vee B) \odot_P C = (A \odot_P C) \vee (B \odot_P C)$

$(iv)(A \wedge B) \odot_P C = (A \odot_P C) \wedge (B \odot_P C)$

The Proof of the following theorem follows from Definition 2.5,Definition 2.6,

Theorem 3.13.If A and B are two PFMs, then

$(i)(A \wedge B) \oplus_P (A \vee B) = A \oplus_P B$

$(ii)(A \wedge B) \odot_P (A \vee B) = A \odot_P B$

Theorem 3.14.If A is any PFM,then

$(i)(A \oplus_P O) = (O \oplus_P A) = A$

$(ii)(A \odot_P J) = (J \odot_P A) = A$

Theorem 3.15.If A is any PFM,then

$(i)(A \oplus_P J) = (J \oplus_P A) = J$

$(ii)(A \odot_P O) = (O \odot_P A) = O$

Theorem 3.16.If A is any PFM,then

$(i) A \oplus_P A \ne J$

$(ii) A \odot_P A \ne O$

IV. RESULTS ON COMPLEMENT OF PYTHAGOREAN FUZZY MATRICES

The operator complement obey the De Morgan's laws for the operators $\oplus_P$ and $\odot_P$. This is established in the following property.

Theorem 4.1.If A and B are two PFMs, then

$(i)(A \oplus_P B)^C = A^C \odot_P B^C$

$(ii)(A \odot_P B)^C = A^C \oplus_P B^C$

$(iii)(A \oplus_P B)^C \le A^C \oplus_P B^C$

$(iv)(A \odot_P B)^C \ge A^C \odot_P B^C$

Proof. The proof (i) and (ii) are trivial.

$(iii)(A \oplus_P B)^C = \left(\left\langle a'_{ij} b'_{ij}, \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} \right\rangle \right)$

$A^C \oplus_P B^C = \left(\left\langle \sqrt{a'^2_{ij} + b'^2_{ij} - a'^2_{ij} b'^2_{ij}}, a_{ij} b_{ij} \right\rangle \right)$

Since $a'_{ij} b'_{ij} \le \sqrt{a'^2_{ij} + b'^2_{ij} - a'^2_{ij} b'^2_{ij}}$

$\sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2} \ge a_{ij} b_{ij}$

Hence $(A \oplus_P B)^C \le A^C \oplus_P B^C$

(iv) It can be proved analogously.

The proof of the following theorem follows from the Definition 2.6.

Theorem 4.2.If A is any PFM,then

$(i) A \odot_P A^C \ge O$

$(ii) A \oplus_P A^C \le J.$

V. THE @ OPERATION ON PYTHAGOREAN FUZZY MATRICES

In this section, we define the @ operation on PFMs and proved their algebraic properties. We discuss the Distributivity laws in the case the operation s of $\oplus_P$, $\odot_P$, $\vee$ and $\wedge$ are combined with each other.

Definition 5.1. Let $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle \right), B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle \right)$ be a two PFMs,the @ operation of PFM is defined by $A@B = \left(\left\langle \sqrt{\frac{a_{ij}^2 + b_{ij}^2}{2}}, \sqrt{\frac{a'^2_{ij} + b'^2_{ij}}{2}} \right\rangle \right)$

Theorem 5.2.If A is a PFMs, then $A@A = A$

Proof.

$$A@A=\left(\left\langle\sqrt{\frac{a_{ij}^2+a_{ij}^2}{2}},\sqrt{\frac{a_{ij}'^2+a_{ij}'^2}{2}}\right\rangle\right)$$
$$=\left(\left\langle\sqrt{\frac{2a_{ij}^2}{2}},\sqrt{\frac{2a_{ij}'^2}{2}}\right\rangle\right)$$
$$=\left(\left\langle a_{ij},a_{ij}'\right\rangle\right)$$
$$=A$$

Theorem 5.3.If A , B and C are three PFMs, then
$(i)A@(B\vee C)=(A@B)\vee(A@C)$
$(ii)A@(B\wedge C)=(A@B)\wedge(A@C)$

Proof.
$(i)(A@B)\vee(A@C)$
$$=\left(\left\langle\begin{array}{l}\sqrt{\frac{1}{2}\max\left\{a_{ij}^2+b_{ij}^2,a_{ij}^2+c_{ij}^2\right\}},\\ \sqrt{\frac{1}{2}\min\left\{a_{ij}'^2+b_{ij}'^2,a_{ij}'^2+c_{ij}'^2\right\}}\end{array}\right\rangle\right)$$
$$=\left(\left\langle\sqrt{\frac{1}{2}(a_{ij}^2+\max\left\{b_{ij}^2,c_{ij}^2\right\})},\sqrt{\frac{1}{2}(a_{ij}'^2+\max\left\{b_{ij}'^2,c_{ij}'^2\right\})}\right\rangle\right)$$
$$=A@(B\vee C)$$

(ii) It can be proved analogously.

The proof of the following theorem follows from Remark 3.7.

Theorem 5.4.If A and B are two PFMs, then
$(i)(A\oplus_P B)\wedge(A@B)=A@B$
$(ii)(A\oplus_P B)\vee(A@B)=A\oplus_P B$
$(iii)(A\odot_P B)\wedge(A@B)=A\odot_P B$
$(iv)(A\odot_P B)\vee(A@B)=A@B$

Proof.
In the following, we shall prove (i),(iii) and (ii),(iv) can be proved analogously.
$(i)(A\oplus_P B)\wedge(A@B)$
$$=\left(\left\langle\begin{array}{l}\min\left\{\sqrt{a_{ij}^2+b_{ij}^2-a_{ij}^2b_{ij}^2},\sqrt{\frac{a_{ij}^2+b_{ij}^2}{2}}\right\},\\ \max\left\{a_{ij}'b_{ij}',\sqrt{\frac{a_{ij}'^2+b_{ij}'^2}{2}}\right\}\end{array}\right\rangle\right)$$
$$=\left(\left\langle\sqrt{\frac{a_{ij}^2+b_{ij}^2}{2}},\sqrt{\frac{a_{ij}'^2+b_{ij}'^2}{2}}\right\rangle\right)$$
$$=A@B$$
$(iii)(A\odot_P B)\wedge(A@B)$
$$=\left(\left\langle\begin{array}{l}\min\left\{a_{ij}b_{ij},\sqrt{\frac{a_{ij}^2+b_{ij}^2}{2}}\right\},\\ \max\left\{\sqrt{a_{ij}'^2+b_{ij}'^2-a_{ij}'^2b_{ij}'^2},\sqrt{\frac{a_{ij}'^2+b_{ij}'^2}{2}}\right\}\end{array}\right\rangle\right)$$
$$=\left(\left\langle a_{ij}b_{ij},\sqrt{a_{ij}'^2+b_{ij}'^2-a_{ij}'^2b_{ij}'^2}\right\rangle\right).$$
$$=A\odot_P B$$

VI. CONCLUSIONS

The set of all PFMs with respect to the operations $\oplus_P$ and $\odot_P$ form a commutative monoid. We proved the De Morgan's laws of PFMs and we discussed the Distributivity laws in the case the operations of $\oplus_P$, $\odot_P$, $\vee$, $\wedge$ are combined each other. Also, we defined the @ operation on PFMs and proved their algebraic properties.

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