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Applications of the Aboodh Transform and the Homotopy Perturbation Method to the Nonlinear Oscillators

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Abstract— In this paper, the differential equation of motion of the classical Helmholtz-Duffing oscillator, Van der Pol, Duffing oscillator and Duffing-Van der Pol oscillator equations have been solved analytically with the help of a new integral transform named Aboodh transform and homotopy perturbation method. By recasting the governing equations as nonlinear eigenvalue problems, we have obtained the excellent approximate analytical solution of the displacement and the relation between amplitude and angular frequency. We have also compared our results with exact numerical results graphically for few cases. Here, we have also demonstrated the sophistication and simplicity of this technique.

Keywords— Aboodh Transform, Homotopy Perturbation Method, Helmholtz-Duffing Oscillator, Van der Pol, Duffing Oscillator, Duffing-Van der Pol Oscillator, Approximate Analytical Solution

I. INTRODUCTION

Many complex problems in nature are due to nonlinear phenomena. Nowadays, nonlinear processes are one of the biggest challenges in finding solutions and are not easy to control, because the nonlinear characteristic of the system abruptly changes due to small changes of valid parameters, including time. Thus, the issue becomes more complicated and, hence, needs an ultimate solution. Therefore, the study of approximate solutions of nonlinear differential equations (NDEs) plays a crucial role in understanding the internal mechanisms of nonlinear phenomena. Advanced nonlinear techniques are significant in solving inherent nonlinear problems, particularly those involving differential equations, dynamical systems, and related areas. In recent years, mathematicians, engineers, and physicists have made significant improvements in finding new mathematical tools related to NDEs and dynamical systems, whose understanding will rely not only on analytical techniques, but also on numerical and asymptotic methods. These professionals have established many effective and powerful methods to handle the NDEs. The study of given nonlinear problems is of crucial importance, not only in all areas of physics, but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by NDEs. Moreover, obtaining exact solutions for nonlinear oscillatory problems has many difficulties. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytical approximation solution than a numerical one for a given nonlinear problem. There are many analytical approaches to solve NDEs.

In this article, we have drawn the attention towards the solution of the differential equations of the nonlinear oscillators as they play an important role in applied mathematics, physics and engineering problems. Also in the theory of harmonics, there are many important phenomena which have practical importance in demonstrating nonlinear effects. In science and engineering, there exist many nonlinear problems, which do not contain any small parameters, especially those with strong nonlinearity. Thus, it is necessary to develop and improve some nonlinear analytical approximations even for large parameters.

The solution to the nonlinear problems are difficult to find and most of them are not exactly solvable. Although the numerical solution to the nonlinear problems is easy, one desires to find the analytical solution to get a better insight of the problem. There are many techniques for solving nonlinear problems such as the harmonic balance method, Krylov-Bogoliubov-Mitropolsky method, weighted linearization method, perturbation procedure for limit cycle analysis, modified Lindstedt-Poincare method, Adomain decomposition method, artificial parameter method, and Nikiforov-Uvarov method [1-9]. Not only these methods have complex calculations, but they fail to handle problems with strong non-linearity.

The homotopy perturbation method (HPM) has been found to be very efficient for solving non-linear equations with known initial or boundary values especially for systems with strong non-linearity in classical and quantum mechanical problems [10-15]. In this method, the solution is given in an infinite series usually converges to an accurate solution. Aboodh introduced a transform derived from the classical Fourier integral for solving ordinary and partial differential equations easily in the time (t) domain [16]. Aboodh transform (AT) has been applied to different types of problems and is found to be very simple but powerful technique [17,18].

In this article, we have applied Aboodh transform based homotopy perturbation method (ATHPM) to solve the nonlinear differential equations to obtain the approximate displacement x and the oscillating frequency ω with high accuracy.

This paper is organized as follows. In section II, we demonstrate briefly the formulation of ATHPM. Applications of ATHPM to nonlinear problems have been shown in section III. Finally, in section IV we provide a brief discussion and our conclusions.

II. Formulation of ATHPM

Aboodh transform is a new transform which is defined for function of exponential order in a set A, where $\{x(t): \exists M, k_1, k_2 > 0 \ni |x(t)| < Me^{-\nu t}dt, t \in (-1)^j \times [0,\infty)\}$ and $x(t)$ is denoted by $A[x(t)] = x(\nu)$ and defined as

$$A[x(t)] = x(\nu) = \frac{1}{\nu}\int_0^\infty x(t)e^{-\nu t}dt, k_1 \le \nu \le k_2 \qquad (1)$$

Some properties of Aboodh Transform which are necessary for our calculations are

$$A[x''(t)] = \nu^2 x(\nu) - \frac{x'(0)}{\nu} - x(0) \qquad (2)$$

$$A[\cos at] = \frac{1}{\nu^2 + a^2}, A[t\sin at] = \frac{2a}{(\nu^2 + a^2)^2} \qquad (3)$$

$$A[t^n] = \frac{n!}{\nu^{n+2}} \qquad (4)$$

Let us consider a nonlinear non-homogeneous differential equation as

$$Lx(t) + \omega^2 x(t) + Rx(t) + Nx(t) = g(t) \qquad (5)$$

with the initial condition $x(0) = x_0(0)$ and $x_0'(0) = 0$. Here, L is the second order linear differential operator $(L \equiv \partial^2/\partial t^2)$, R is the linear operator having an order less than L, N is the nonlinear operator, $g(t)$ is the non-homogeneous term and ω^2 is any parameter.

Now, taking the Aboodh Transform on both sides of (5) we get

$$A[Lx(t)] + \omega^2 A[x(t)] + A[Rx(t)] + A[Nx(t)] = A[g(t)] \qquad (6)$$

Using the differential properties of the Aboodh Transform as mentioned above and the initial conditions (6) can be written

$$\begin{aligned} x(\nu) &= \left(\frac{1}{\nu^2+\omega^2}\right)x(0) + \frac{x'(0)}{\nu(\nu^2+\omega^2)} \\ &- \left\{\left(\frac{1}{\nu^2+\omega^2}\right)A[Rx(t)] + \left(\frac{1}{\nu^2+\omega^2}\right)A[Nx(t)]\right. \\ &\left. + \left(\frac{1}{\nu^2+\omega^2}\right)A[g(t)]\right\} \end{aligned} \qquad (7)$$

Taking Inverse Aboodh Transform on both sides of (7) leads to

$$\begin{aligned} x(t) &= X_0(t) - \left\{A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[Rx(t)]\right]\right. \\ &\left. + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[Nx(t)]\right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[g(t)]\right]\right. \end{aligned} \qquad (8)$$

where $X_0(t) = \left(\frac{1}{\nu^2+\omega^2}\right)x(0) + \frac{x'(0)}{\nu(\nu^2+\omega^2)}$.

According to the homotopy perturbation method, we can write $x(t) = \sum_{n=0}^{\infty} p^n x_n(t)$ and the nonlinear term as $Nx(t) = \sum_{n=0}^{\infty} p^n H_n(x)$

where He's polynomial $H_n(x)$ can be written as

$$H_n(x) = \frac{1}{n!}\frac{d^n}{dp^n}\left[N\left(\sum_{n=0}^{\infty} p^n x_n(t)\right)\right]_{p=0}, n = 0,1,2,3\ldots \qquad (9)$$

Applying HPM and substituting the value of $x(t)$ and $Nx(t)$ in (8), we get

$$\sum_{n=0}^{\infty} p^n u_n(t) = U_0(t) - p\left\{ A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[R\sum_{n=0}^{\infty} p^n u_n(t) \right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[\sum_{n=0}^{\infty} p^n H_n(t) \right] \right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[g(t)] \right] \right\} \quad (10)$$

Equation (10) is the coupling between the Aboodh Transform and the homotopy perturbation method using He's polynomials where p is an imbedding parameter. Comparing the coefficient of like power of p, we get from (10) the following equations

$$p^0 : x_0(t) = X_0(t) \quad (11)$$

$$p^1 : x_1(t) = -\left\{ A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[Rx_0(t)] \right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[H_0(x_0(t))] \right] \right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[g(t)] \right] \quad (12)$$

$$p^2 : x_2(t) = -\left\{ A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[Rx_1(t)] \right] + A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[H_1(x_0(t), x_1(t)] \right] \right\} \quad (13)$$

The approximate solution is

$$x_{ATHPM}(t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n x_n(t) = x_0(t) + x_1(t) + x_2(t) + \ldots \quad (14)$$

III. Applications

In order to assess the accuracy of the ATHPM which has been presented in section II, it is applied to different types of nonlinear oscillators and the results are compared with the exact results.

Helmholtz-Duffing Oscillator

The common form of the differential equation of motion of a Helmholtz-Duffing oscillator is given as

$$\frac{d^2x}{dt^2} + x + (1-\alpha)x^2 + \alpha x^3 = 0 \quad (15)$$

Here we consider the boundary conditions at *t=0*, $x(0) = a, x'(0) = 0$. Now, for our purpose we rewrite (15) as

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(\omega^2 - 1\right)x + (\alpha-1)x^2 - \alpha x^3 \quad (16)$$

Using Aboodh Transform and applying the boundary conditions to (16) we get

$$x(\nu) = \left(\frac{1}{\nu^2+\omega^2} \right) a + (\omega^2-1)\left(\frac{1}{\nu^2+\omega^2} \right) A[x] - (\alpha-1)\left(\frac{1}{\nu^2+\omega^2} \right) A[x^2] - \alpha\left(\frac{1}{\nu^2+\omega^2} \right) A[x^3] \quad (17)$$

Taking the inverse Aboodh Transform on both sides of (17) we get

$$x(t) = a\cos\omega t + (\omega^2-1)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[x] \right] + (\alpha-1)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[x^2] \right] - \alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[x^3] \right] \quad (18)$$

Applying the HPM, we can write (18) as

$$\sum_{p=0}^{\infty} p^n x_n(t) = a\cos\omega t + p\left\{ (\omega^2-1)\, A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[\sum\nolimits_{p=0}^{\infty} p^n x_n(t) \right] \right] + (\alpha-1)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[\left(\sum\nolimits_{p=0}^{\infty} p^n x_n(t) \right)^2 \right] \right] - \alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[\left(\sum\nolimits_{p=0}^{\infty} p^n x_n(t) \right)^3 \right] \right] \quad (19)$$

Equating the coefficient of p^0 and p^1, we obtain from (19)

$$p^0 : x_0(t) = a\cos\omega t \quad (20)$$

$$p^1 : x_1(t) = \left(\omega^2-1\right) A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A[x_0] \right] - \left(\alpha-1\right) A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[x_0^2 \right] \right] - \alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2} \right) A\left[x_0^3 \right] \right] \quad (21)$$

After some mathematical calculation of inverse Aboodh Transform, we get from (21)

$$x_1(t)=\frac{1}{2\omega}\left(a\left(\omega^2-1\right)+\frac{3\alpha a^3}{4}\right)t\sin\omega t$$
$$+\frac{a^2(\alpha-1)}{2\omega^2}\left(1-\cos\omega t\right)+\frac{a^2(\alpha-1)}{6\omega^2}\left(\cos\omega t\right. \quad (22)$$
$$\left.-\cos 2\omega t\right)-\frac{\alpha a^3}{32\omega^2}\left(\cos\omega t-\cos 3\omega t\right)$$

Here, the term $t\sin\omega t$ is a secular term which must be absent if and only if $\frac{1}{2\omega}\left(a\left(\omega^2-1\right)+\frac{3\alpha a^3}{4}\right)=0$. Hence, angular frequency of oscillation is $\omega=1+{}^{3\alpha a^2}\!/_8$ and time period $T=2\pi\left(1+{}^{3\alpha a^2}\!/_8\right)$. So, we can write the approximate solution obtained from (20) and (22) as

$$x_{ATHPM}(t)=a\cos\omega t+\frac{a^2(\alpha-1)}{2\omega^2}\left(1-\cos\omega t\right)$$
$$+\frac{a^2(\alpha-1)}{6\omega^2}\left(\cos\omega t-\cos 2\omega t\right) \quad (23)$$
$$+\frac{\alpha a^3}{32\omega^2}\left(\cos\omega t-\cos 3\omega t\right)$$

Duffing Oscillator

Here we consider a damping Duffing oscillator like [19]

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}+x^3=0, x(0)=1, x'(0)=1 \quad (24)$$

Applying Aboodh Transform to (24), we get

$$x(v)=\frac{1}{v^2}+\frac{1}{v^3}-\frac{1}{v^2}A\left[\frac{dx}{dt}\right]-\frac{1}{v^2}A\left[x^3\right] \quad (25)$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (25)

$$x(t)=1+t-A^{-1}\left[\frac{1}{v^2}A\left[\frac{dx}{dt}\right]\right]-A^{-1}\left[\frac{1}{v^2}A\left[x^3\right]\right] \quad (26)$$

Applying HPM to (26) we can write

$$\sum_{p=0}^{\infty}p^n x_n(t)=1+t-p\left\{A^{-1}\left[\frac{1}{v^2}A\left[\frac{d\sum_{p=0}^{\infty}p^n x_n(t)}{dt}\right]\right]\right. \quad (27)$$
$$\left.+A^{-1}\left[\frac{1}{v^2}A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^3\right]\right]\right\}$$

Equating the coefficient of p^0 and p^1 from (27) we obtain

$$p^0: x_0(t)=1+t$$
$$p^1: x_1(t)=-A^{-1}\left[\frac{1}{v^2}A\left[\frac{dx_0}{dt}\right]\right]-A^{-1}\left[\frac{1}{v^2}A\left[x_0^3\right]\right] \quad (28)$$

After some mathematical calculation of inverse Aboodh Transform we get from (28) as

$$x_1(t)=-t^2-\frac{1}{2}t^3-\frac{1}{4}t^4-\frac{1}{20}t^5 \quad (29)$$

So, the approximate solution of (24) up to first order correction is obtained from (27) and (29) as

$$x_{ATHPM}(t)=1+t-t^2-\frac{1}{2}t^3-\frac{1}{4}t^4-\frac{1}{20}t^5 \quad (30)$$

Duffing Oscillator

Let us consider another damping Duffing oscillator like [20]

$$\frac{d^2x}{dt^2}+x+x^3=0, x(0)=1, x'(0)=5 \quad (31)$$

Applying Aboodh Transform to (31) we get

$$x(v)=\frac{1}{v^2}+\frac{5}{v^3}-\frac{1}{v^2}A\left[x\right]-\frac{1}{v^2}A\left[x^3\right] \quad (32)$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (32)

$$x(t)=1+5t-A^{-1}\left[\frac{1}{v^2}A\left[x\right]\right]-A^{-1}\left[\frac{1}{v^2}A\left[x^3\right]\right] \quad (33)$$

Applying HPM to (33) we get

$$\sum_{p=0}^{\infty}p^n x_n(t)=1+5t-p\left\{A^{-1}\left[\frac{1}{v^2}A\left[\sum_{p=0}^{\infty}p^n x_n(t)\right]\right]\right. \quad (34)$$
$$\left.+A^{-1}\left[\frac{1}{v^2}A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^3\right]\right]\right\}$$

Equating the coefficient of p^0 and p^1 from (34) we obtain

$$p^0 : x_0(t) = 1 + 5t$$
$$p^1 : x_1(t) = -A^{-1}\left[\frac{1}{v^2} A\left[x_0\right]\right] - A^{-1}\left[\frac{1}{v^2} A\left[x_0^3\right]\right] \tag{35}$$

After some mathematical calculation of inverse Aboodh Transform we get from (35) as

$$x_1(t) = -t^2 - \frac{10}{3}t^3 - \frac{25}{4}t^4 - \frac{25}{4}t^5 \tag{36}$$

So, the solution of (31) up to first order correction is

$$x_{ATHPM}(t) = 1 + 5t - t^2 - \frac{10}{3}t^3 - \frac{25}{4}t^4 - \frac{25}{4}t^5 \tag{37}$$

Duffing-Van der Pol Oscillator

Let us consider a Duffing-Van der Pol oscillator as

$$\frac{d^2x}{dt^2} + x + \alpha\frac{dx}{dt} + x^3 = 0, x(0) = 1, x'(0) = \alpha\omega \tag{38}$$

Now, we rewrite (38) as

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(\omega^2 - 1\right)x - \alpha\frac{dx}{dt} - x^3 \tag{39}$$

Applying Aboodh Transform to (39) we get

$$x(v) = \left(\frac{\omega}{v\left(v^2+\omega^2\right)}\right)a + (\omega^2-1)\left(\frac{1}{v^2+\omega^2}\right)A[x] - \alpha\left(\frac{1}{v^2+\omega^2}\right)A\left[\frac{dx}{dt}\right] - \left(\frac{1}{v^2+\omega^2}\right)A[x^3] \tag{40}$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (40)

$$x(t) = a\sin\omega t + (\omega^2-1)A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[x]\right] - \alpha A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\frac{dx}{dt}\right]\right] - A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[x^3]\right] \tag{41}$$

Applying HPM to (41) we obtain

$$\sum_{p=0}^{\infty} p^n x_n(t) = a\sin\omega t + p\left\{(\omega^2-1)A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right) A\left[\sum_{p=0}^{\infty} p^n x_n(t)\right]\right] - \alpha A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right) A\left[\frac{d\sum_{p=0}^{\infty} p^n x_n(t)}{dt}\right]\right] - A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right) A\left[\left(\sum_{p=0}^{\infty} p^n x_n(t)\right)^3\right]\right]\right\} \tag{42}$$

Equating the coefficient of p^0 and p^1 from (42) we obtain

$$p^0 : x_0(t) = a\sin\omega t \tag{43}$$

$$p^1 : x_1(t) = \left(\omega^2-1\right)A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A[x_0]\right] + \alpha A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[\frac{dx_0}{dt}\right]\right] - A^{-1}\left[\left(\frac{1}{v^2+\omega^2}\right)A\left[x_0^3\right]\right] \tag{44}$$

After some mathematical calculation of inverse Aboodh Transform and eliminating the secular term, we get the first order correction term as

$$x_1(t) = -\frac{\alpha}{2}ta\sin\omega t + \frac{3a^3}{32\omega^2}\left(\sin\omega t - \frac{1}{3}\sin 3\omega t\right) \tag{45}$$

with angular frequency $\omega = 1 + \frac{3}{8}a^2$ and time period $T = 2\pi/\left(1+\frac{3}{8}a^2\right)$. So, we can write the approximate solution of (38) up to first order correction as

$$x_{ATHPM}(t) = ae^{-\frac{at}{2}}\sin\omega t + \frac{3a^3}{32\omega^2}\left(\sin\omega t - \frac{1}{3}\sin 3\omega t\right) \tag{46}$$

Duffing-Van der Pol Oscillator

The classical Duffing-Van der Pol oscillator appears in many physical problems and is governed by the following nonlinear differential equation like [21]

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x + \alpha x^3 = 0, x(0) = a, x'(0) = 0 \tag{47}$$

Now, we rewrite (47) as

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(\omega^2-1\right)x + \mu(1-x^2)\frac{dx}{dt} - \alpha x^3 \tag{48}$$

Applying Aboodh Transform to (48) we get

$$x(\nu)=\left(\frac{1}{\nu^2+\omega^2}\right)a+(\omega^2-1)\left(\frac{1}{\nu^2+\omega^2}\right)A[x]$$
$$+\mu\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{dx}{dt}\right]-\mu\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^2\frac{dx}{dt}\right] \tag{49}$$
$$-\alpha\left(\frac{1}{\nu^2+\omega^2}\right)A[x^3]$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (49)

$$x(\nu)=a\cos\omega t+(\omega^2-1)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x]\right]$$
$$+\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{dx}{dt}\right]\right]-\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right. \tag{50}$$
$$\left.A\left[x^2\frac{dx}{dt}\right]\right]-\alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x^3]\right]$$

Applying HPM to (50) we obtain

$$\sum_{p=0}^{\infty}p^n x_n(t)=a\cos\omega t+p\left\{(\omega^2-1)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\right.$$
$$\left.A\left[\sum_{p=0}^{\infty}p^n x_n(t)\right]\right]+\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{d\sum_{p=0}^{\infty}p^n x_n(t)}{dt}\right]\right]$$
$$-\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^2\frac{d\sum_{p=0}^{\infty}p^n x_n(t)}{dt}\right]\right] \tag{51}$$
$$\left.-\alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^3\right]\right]\right\}$$

Equating the coefficient of p^0 and p^1 from (51) we obtain

$$p^0: x_0(t)=a\cos\omega t \tag{52}$$

$$p^1: x_1(t)=\left(\omega^2-1\right)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x_0]\right]$$
$$+\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{dx_0}{dt}\right]\right]-\mu A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right. \tag{53}$$
$$\left.A\left[x_0^2\frac{dx_0}{dt}\right]\right]-\alpha A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x_0^3\right]\right]$$

After some mathematical calculation of inverse Aboodh Transform and eliminating the secular term, we get the first order correction term as

$$x_1(t)=\frac{3a^3}{32\omega}\mu\left(\sin\omega t-\frac{1}{3}\sin 3\omega t\right)-\frac{\alpha a^3}{32\omega^2}\left(\cos\omega t-\cos 3\omega t\right) \tag{54}$$

With the amplitude $a=2$, angular frequency $\omega=1+\tfrac{3}{4}\alpha a^2$ and time period $T=2\pi/\left(1+\tfrac{3}{4}\alpha a^2\right)$. So, we can write the approximate solution of (47) up to first order correction as

$$x_{ATHPM}(t)=a\cos\omega t+\frac{3a^3}{32\omega}\mu\left(\sin\omega t-\frac{1}{3}\sin 3\omega t\right)$$
$$-\frac{\alpha a^3}{32\omega^2}\left(\cos\omega t-\cos 3\omega t\right) \tag{55}$$

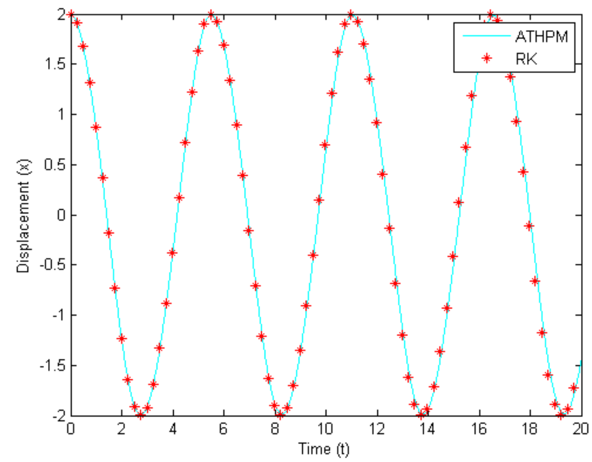

Figure 1. Time(t) vs displacement(x) curves obtained from numerical (RK) and ATHMP with a = 2, μ = 0.1 and α = 0.01

We have plotted the displacement x(t) from numerical solution for a = 2, μ = 0.1, α = 0.01 and compared the same obtained from Runge-Kutta (RK) calculations. It is found that the displacement obtained from RK and ATHPM are matching very closely.

When $\alpha=0$ we get equation of motion of nonlinear oscillator as

$$\frac{d^2x}{dt^2}-\mu\left(1-x^2\right)\frac{dx}{dt}+x=0, x(0)=a, x'(0)=0 \tag{56}$$

and we obtain the approximate solution from (55) as

$$x_{ATHPM}(t)=a\cos\omega t+\frac{3a^3}{32\omega}\mu\left(\sin\omega t-\frac{1}{3}\sin 3\omega t\right) \tag{57}$$

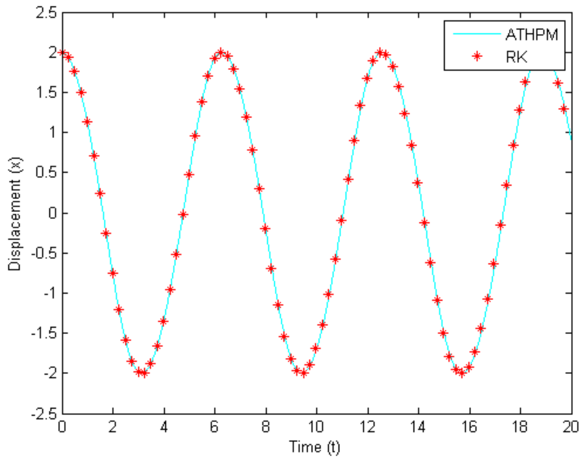

Figure 2. Time(t) vs displacement(x) curves obtained from numerical (RK) and ATHMP with a = 2, μ = 0.1 and α = 0

We have plotted the displacement x(t) from numerical solution for a = 2, μ = 0.1, α = 0 and compared the same obtained from Runge-Kutta (RK) calculations. It is found that the displacement obtained from RK and ATHPM are matching very closely.

Classical Fractional Van der Pol Oscillator

Consider the classical fractional Van der Pol damped nonlinear oscillator as [22]

$$\frac{d^2x}{dt^2}+x^{1/3}+\varepsilon(1-x^2)\frac{dx}{dt}=0, x(0)=a, x'(0)=0 \tag{58}$$

Now, we rewrite (58) as

$$\frac{d^2x}{dt^2}+\omega^2 x=\omega^2 x-x^{1/3}-\varepsilon\frac{dx}{dt}+\varepsilon x^2\frac{dx}{dt} \tag{59}$$

Applying Aboodh Transform to (59) we get

$$\begin{aligned}x(\nu)=&\left(\frac{1}{\nu^2+\omega^2}\right)a+\omega^2\left(\frac{1}{\nu^2+\omega^2}\right)A[x]\\ &-\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^{1/3}\right]-\varepsilon\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{dx}{dt}\right]\\ &+\varepsilon\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^2\frac{dx}{dt}\right]\end{aligned} \tag{60}$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (60)

$$\begin{aligned}x(t)=&a\cos\omega t+\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x]\right]\\ &-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^{1/3}\right]\right]-\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\\ &\left.A\left[\frac{dx}{dt}\right]\right]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^2\frac{dx}{dt}\right]\right]\end{aligned} \tag{61}$$

Applying HPM to (61) we obtain

$$\begin{aligned}\sum_{p=0}^{\infty}p^n x_n(t)=&a\cos\omega t+p\left\{\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\right.\\ &A\left[\sum_{p=0}^{\infty}p^n x_n(t)\right]\Bigg]-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\\ &A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^{1/3}\right]\Bigg]-\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\\ &A\left[\frac{d\sum_{p=0}^{\infty}p^n x_n(t)}{dt}\right]\Bigg]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\\ &A\left[\left(\sum_{p=0}^{\infty}p^n x_n(t)\right)^2\frac{d\sum_{p=0}^{\infty}p^n x_n(t)}{dt}\right]\Bigg]\Bigg\}\end{aligned} \tag{62}$$

Equating the coefficient of p^0 and p^1 from (62) we obtain

$$p^0: x_0(t)=a\cos\omega t \tag{63}$$

$$\begin{aligned}p^1: x_1(t)=&\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x_0]\right]\\ &-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x_0^{1/3}\right]\right]-\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\\ &\left.A\left[\frac{dx_0}{dt}\right]\right]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x_0^2\frac{dx_0}{dt}\right]\right]\end{aligned} \tag{64}$$

The Fourier series for $(\cos\omega t)^{1/3}$ has been calculated and is given by $(\cos\omega t)^{1/3}=b_1\cos\omega t+b_2\cos 3\omega t+\cdots$ where $b_1=1.15960, b_2=-0.231919$. With the help of (63) we get from (64)

$$x_1(t)=\left(a\omega^2-a^{1/3}b_1\right)A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\cos\omega t\right]\right]$$
$$a^{1/3}b_2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\cos3\omega t\right]\right]+\left(a\varepsilon\omega-\frac{\varepsilon\omega a^3}{4}\right)$$
$$A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\sin\omega t\right]\right]-\frac{\varepsilon\omega a^3}{4}$$
$$A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\sin3\omega t\right]\right] \quad (65)$$

After some mathematical calculation of inverse Aboodh Transform and avoiding the secular terms, by putting $\left(a\omega^2-a^{1/3}b_1\right)=0$ and $\left(a\varepsilon\omega-\varepsilon\omega a^3/4\right)=0$, we obtain the amplitude $a=2$ and the angular frequency $\omega=\sqrt{b_1}/a^{1/3}=0.8547$ which is same as obtained by the iteration procedure [24]. Hence, the approximate periodic solution takes the form $x_{app}(t)=2\cos(0.8547t)$. The exact solution for the classical fractional Van der Pol damped nonlinear oscillator is $x_{ex}=\frac{2a}{\left(a^2+\left(4-a^2\right)e^{-\varepsilon t}\right)^{1/2}}\cos\left(\omega_{ex}t\right)$ where, $\omega_{ex}=\frac{\sqrt{\pi}}{\sqrt{8/3}}\frac{\Gamma(5/4)}{a^{1/3}\Gamma(7/4)}$ [25]. We get the approximate solution of (58) up to first order correction as

$$x_{ATHPM}(t)=a\cos\omega t+\frac{a^{1/3}b_2}{8\omega^2}\left(\cos3\omega t-\cos\omega t\right)$$
$$-\frac{3\varepsilon a^3}{32\omega}\left(\sin\omega t-\frac{1}{3}\sin3\omega t\right) \quad (66)$$

Rayleigh Equation

The special case of the fractional Van der Pol damped nonlinear oscillator or Rayleigh equation can be represented by [26]

$$\frac{d^2x}{dt^2}+x^3-\varepsilon\left(\frac{dx}{dt}-\frac{1}{3}\left(\frac{dx}{dt}\right)^3\right)=0, x(0)=a, x'(0)=0 \quad (67)$$

Now, we rewrite (67) as

$$\frac{d^2x}{dt^2}+\omega^2x=\omega^2x-x^3+\varepsilon\frac{dx}{dt}-\frac{\varepsilon}{3}\left(\frac{dx}{dt}\right)^3 \quad (68)$$

Applying Aboodh Transform to (68) we get

$$x(\nu)=\left(\frac{1}{\nu^2+\omega^2}\right)a+\omega^2\left(\frac{1}{\nu^2+\omega^2}\right)A[x]$$
$$-\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^3\right]+\varepsilon\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\frac{dx}{dt}\right]$$
$$-\frac{\varepsilon}{3}\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\left(\frac{dx}{dt}\right)^3\right] \quad (69)$$

Taking inverse Aboodh Transform and applying the boundary conditions we get from (69)

$$x(t)=a\cos\omega t+\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x]\right]$$
$$-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x^3\right]\right]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.$$
$$\left.A\left[\frac{dx}{dt}\right]\right]-\frac{\varepsilon}{3}A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\left(\frac{dx}{dt}\right)^3\right]\right] \quad (70)$$

Applying HPM to (70) we obtain

$$\sum_{p=0}^{\infty}p^nx_n(t)=a\cos\omega t+p\left\{\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.\right.$$
$$\left.A\left[\sum_{p=0}^{\infty}p^nx_n(t)\right]\right]-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.$$
$$\left.A\left[\left(\sum_{p=0}^{\infty}p^nx_n(t)\right)^3\right]\right]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.$$
$$\left.A\left[\frac{d\sum_{p=0}^{\infty}p^nx_n(t)}{dt}\right]\right]-\frac{\varepsilon}{3}A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.$$
$$\left.\left.A\left[\left(\frac{d\sum_{p=0}^{\infty}p^nx_n(t)}{dt}\right)^3\right]\right]\right\} \quad (71)$$

Equating the coefficient of p^0 and p^1 from (71) we obtain

$$p^0: x_0(t)=a\cos\omega t \quad (72)$$

$$p^1: x_1(t)=\omega^2A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A[x_0]\right]$$
$$-A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[x_0^3\right]\right]+\varepsilon A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)\right.$$
$$\left.A\left[\frac{dx_0}{dt}\right]\right]-\frac{\varepsilon}{3}A^{-1}\left[\left(\frac{1}{\nu^2+\omega^2}\right)A\left[\left(\frac{dx_0}{dt}\right)^3\right]\right] \quad (73)$$

Proceeding in the same way as before and avoiding the secular terms, $a\omega^2 - \frac{3}{4}a^3 = 0$ and $1 - \frac{a^2}{4}\omega^2 = 0$, we get the approximate solution up to the first order correction as

$$x_{ATHPM}(t) = a\cos\omega t + \frac{a^3}{32\omega^2}\left(\cos 3\omega t - \cos\omega t\right) - \frac{\varepsilon a^3}{32\omega}\left(\sin\omega t - \frac{1}{3}\sin 3\omega t\right) \quad (74)$$

It is found that $a = 1.51967$ and $\omega = 1.31607$. The exact periodic solution is

$$x_{ex} = \frac{2a}{\left(a^2\omega_{ex}^2 + \left(4 - a^2\omega_{ex}^2\right)e^{-\varepsilon t}\right)^{\frac{1}{2}}}\cos\left(\omega_{ex}t\right)$$

where $\omega_{ex} = \frac{\sqrt{\pi}}{\sqrt{8}}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$ [25].

IV. CONCLUSION

We have applied a simple perturbation theory ATHPM to solve the nonlinear differential equation of motion for nonlinear oscillators. ATHPM is found to give analytic solutions with all perturbative corrections to both the displacement and the oscillation frequency in a very simple and straightforward manner. Here, we attain added realism and sophistication of this ATPHM by dealing with the differential equation of motion of nonlinear oscillators to obtain an analytical expression for the frequency of oscillation and the displacement. It is shown that the solution converges very fast. Even first order correction is sufficient for getting accurate results. This method not only gives very accurate numerical values of displacement but also gives an idea about the contributions from different harmonics to it. We may conclude that this technique is not only simple but also elegant way to study a wide class of realistic non-exactly solvable problems.

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